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To cite this article: Marek Mozrzymas *et al* 2018 *New J. Phys.* **20** 053006

IOP Publishing New J. Phys. 20 (2018) 053006 https://doi.org/10.1088/1367-2630/aab8e7

New Journal of Physics

The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark

OPEN ACCESS

RECEIVED
20 August 2017

REVISED
30 January 2018

ACCEPTED FOR PUBLICATION
22 March 2018

PUBLISHED
2 May 2018

PAPER

Optimal port-based teleportation

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Keywords: quantum information, representation theory, teleportation

Abstract

Deterministic port-based teleportation (dPBT) protocol is a scheme where a quantum state is guaranteed to be transferred to another system without unitary correction. We characterise the best achievable performance of the dPBT when both the resource state and the measurement is optimised. Surprisingly, the best possible fidelity for an arbitrary number of ports and dimension of the teleported state is given by the largest eigenvalue of a particular matrix—Teleportation Matrix. It encodes the relationship between a certain set of Young diagrams and emerges as the optimal solution to the relevant semidefinite programme.

1. Introduction

Quantum teleportation is one of the earliest and most widely used primitives in Quantum Information Science which performs an arbitrary quantum state transfer between two spatially separated systems [2]. It involves pre-sharing an entangled resource state and consists of three simple stages. The first stage involves a joint measurement of the teleported subsystem together with the share of the resource state on the sender's side. In the second step, classical measurement outcome is communicated to the receiver. The last step consists of applying a requisite correction operation which recovers the transmitted quantum state.

Port-based teleportation (PBT) discovered by Ishizaka and Hiroshima [7] is a particular teleportation protocol which stands out for its simplicity and surprising qualities which are unattainable by the pre-existing set of protocols. They were able to reduce the three-step procedure to the one where the remaining correction step is trivial. In this protocol, the sender and the receiver share a large entangled resource state and the sender implements a joint POVM on the teleported system and the resource state. Depending on the type of POVM, one distinguishes two operational regimes: probabilistic and deterministic. In the former case, which is well-understood only when one teleports qubits, the measurement is designed to ensure that the teleported state arrives intact to the receiver, but there is a small probability of failure. In the latter case, the state always gets to the receiver but incurs some distortion. In both protocols the sender communicates the classical measurement outcome (including the failure in the former case) to the receiver who then traces out part of the resource state indicated by the classical communication and finishing with the teleported state in the case of deterministic port-based teleportation (dPBT) or maximally mixed state in case of the probabilistic PBT.

While the optimal functioning of the probabilistic PBT is well-understood, for a number of practical applications it may be critical to have a teleportation protocol without a unitary correction which always succeeds even when the replica is distorted. Understanding the feasibility of such protocols (with optimal measurements and the corresponding resource state) for an arbitrary number of ports and local dimension of the teleported state remained a difficult open problem.

Despite the superficial similarity to the probabilistic PBT, characterising the optimal performance of the dPBT remained elusive due to the distortion which affected the teleported state—the existing tools were ill-suited for the analysis of the resulting quantum state on the receiver. In our work, we show that the optimal performance regime for the dPBT, remarkably, can be reduced to the study of a static object—Teleportation Matrix (TM). This extraordinarily simple matrix emerges as a result of an SDP optimisation and characterises the abstract relationship between the input and the output states of the protocol.

In this work, we obtain a relationship between the dPBT and its companion TM and provide a convergent algorithm to determine its infinity norm that characterises the best possible fidelity of teleportation when both the resource state and measurement are optimised. In particular, when the dimension of the teleported state is greater or equal to the number of ports, the maximal eigenvalue is obtained analytically. In the other case, we provide a convergent algorithm to compute it.

In section 2 we review the connection of PBT protocols with the algebra of partially transposed permutation operators, followed by a short review of basic facts about the induced and restricted representations of the symmetric group $S(N)$ in section 3. In the same section, we also prove a group-theoretic lemma about characters of the induced representations which will play an important role in the following sections. Then, in the first part of section 4, we formally introduce the TM and study its properties. In particular, we present an analytical expression for its eigenvalues and corresponding eigenvector when the dimension of underlying local Hilbert space is large enough compared to the number of ports. In the second part, we provide an alternative approach to computing spectral properties of the TM. Finally, in section 5 we show how it naturally appears as a result of semidefinite optimization and describe a convergent algorithm which calculates its infinity norm with the corresponding eigenvector when the dimension of the local Hilbert space is smaller than a number of ports.

2. The dPBT protocol and its connection to a representation of the algebra

We now recall the details of the dPBT introduced in [6–8], and introduce the notation emphasise the connection with the algebra of partially transposed permutation operators $\mathcal{A}_n^{t_n}$. Here we review the most important facts regarding the representation of $\mathcal{A}_n^{t_n}(d)$ (for detailed discussion of properties of $\mathcal{A}_n^{t_n}(d)$ see [11–13]). In dPBT, two parties, Alice and Bob, share a resource state consisting of N copies of bipartite maximally entangled states $|\psi^+\rangle$. Then Alice performs a joint measurement on her half of the resource state and the unknown state θ_C which she wants to teleport by performing a POVM $\{\widetilde{\Pi}_a\}_{a=1}^N$, where each $\widetilde{\Pi}_a$ is a square root measurement [7]. She then communicates the measurement outcome $a\in\{1,\ldots,N\}$ to Bob. This outcome a labels the port on Bob's side which contains the teleported state. Bob then traces out all the ports except for the ath. In this protocol, teleportation always succeeds but the teleported state arrives distorted. To characterise the performance of the dPBT we need to evaluate the fidelity of teleportation F [7]:

$$F = \frac{1}{d^2}\sum_{a=1}^N \mathrm{Tr}[\sigma_a\widetilde{\Pi}_a] = \frac{1}{d^2}\sum_{a=1}^N \mathrm{Tr}[\sigma_a\rho^{-1/2}\sigma_a\rho^{-1/2}],\quad \widetilde{\Pi}_a = \rho^{-1/2}\sigma_a\rho^{-1/2}, \tag{1}$$

which is a function of a number of ports N and local dimension of the Hilbert space d. For $1 \leqslant a \leqslant N$

$$\sigma_a = \frac{1}{d^N}\mathbf{1}_{\overline{aC}}\otimes \widetilde{P}^+_{aC} = \frac{1}{d^N}\mathbf{1}_{\overline{aC}}\otimes V^{t_C}(a, C), \tag{2}$$

where $\mathbf{1}_{\overline{aC}}$ denotes the identity operator acting on all subsystems except ath Cth, $\widetilde{P}^+_{aC}$ denotes an unnormalised projector onto the maximally entangled state $|\Phi^+\rangle_{aC} = \frac{1}{\sqrt{d}}\sum_{i=1}^d|ii\rangle_{aC}$ between subsystems a and C, where the set $\{|i\rangle\}_{i=1}^d$ is the standard basis in $\mathbb{C}^d$. In the second equality in (2) we use a well-known fact that $\widetilde{P}^+_{aC} = V^{t_C}(a, C)$, where t_C is a partial transposition with respect to subsystem C performed on permutation operator $V(a, C)$ acting between subsystems a and C. The operator ρ in (1) is called the PBT operator, and can be expressed as (see [12]):

$$\rho = \sum_{a=1}^N \sigma_a = \frac{1}{d^N}\sum_{a=1}^N \mathbf{1}_{\overline{aC}}\otimes V^{t_C}(a, C) = \frac{1}{d^N}\eta. \tag{3}$$

Since every element $\mathbf{1}_{\overline{aC}}\otimes V(a, C)$ acts as a permutation on the full Hilbert space $(\mathbb{C}^d)^{\otimes n}$, where $n = N+1$, we will further denote it by $V(a, C)$. To keep the notation consistent with the earlier works that study $\mathcal{A}_n^{t_n}(d)$ we label subsystem C by the index n, then expressions (2) and (3) read

$$\sigma_a = \frac{1}{d^N}V^{t_n}(a, n),\quad \rho = \sum_{a=1}^N\sigma_a = \frac{1}{d^N}\sum_{a=1}^N V^{t_n}(a, n) = \frac{1}{d^N}\eta. \tag{4}$$

From the above identities, it follows that ρ is closely related to algebra $\mathcal{A}_n^{t_n}(d)$ of partially transposed permutation operators where partial transposition t_n is performed with respect to the nth subsystem. The operator ρ can be regarded as an element of the algebra $\mathcal{A}_n^{t_n}(d)$. From [11, 13] we know that the full algebra $\mathcal{A}_n^{t_n}(d)$ splits into a direct sum of two left ideals $\mathcal{A}_n^{t_n}(d) = \mathcal{M}\oplus\mathcal{S}$. From [12] we also know that the part of the algebra $\mathcal{A}_n^{t_n}(d)$ containing the ideal $\mathcal{S}$ does not play any role in the description of the dPBT, so we will not discuss it here. In the ideal $\mathcal{M}$, all irreducible representations (irreps) of $\mathcal{A}_n^{t_n}(d)$ are labelled by the irreps of the symmetric group $S(N-1)$, and they related to the irreps of the group $S(N)$ induced by those irreps of $S(N-1)$.

Furthermore, we denote the corresponding projector (including multiplicities) on a chosen irrep labelled by $\alpha \vdash N-1$ (the symbol $\vdash$ indicates that the diagram α is obtained for $N-1$ boxes) by M_α, and its support space by $S(M_\alpha)$. Further by P_μ we denote the Young projector (including multiplicities) onto irrep of $S(N)$ labelled by $\mu \vdash N$ induced from a given irrep α of $S(N-1)$. It occurs when a Young diagram $\mu \vdash N$ can be obtained from a Young diagram $\alpha \vdash N-1$ by adding a single box $\square$ (we denote this by $\mu \in \alpha$), and when all irreps labelled by α and μ occur. The latter happens when the height of the first column of α and μ is less or equal to the dimension d of the local Hilbert space (i.e. when $h(\alpha) \leqslant d, h(\mu) \leqslant d$). Define projectors

$$\forall \ \mu \in \alpha \quad F_\mu(\alpha) \equiv M_\alpha P_\mu, \tag{5}$$

which project onto irreps of $S(N)$ contained in M_α labelled by Young diagrams μ and induced from the irreps of $S(N-1)$ labelled by α [12]. Denoting by P_α a Young projector onto irrep labelled by $\alpha \vdash N-1$ we get the following representation of η from equation (4):

$$\eta = \sum_\alpha \eta(\alpha) = \sum_\alpha V(a, N) P_\alpha V^{t_n}(N, n) V(a, N). \tag{6}$$

The support of every $\eta(\alpha)$ is the space $S(M_\alpha)$ which is invariant with respect to the action of $S(n-1)$, so we see that $F_\mu(\alpha)$ are eigenprojectors of $\eta(\alpha)$. From [12] we know that projectors $F_\mu(\alpha)$ can be written as:

$$F_\mu(\alpha) = \gamma_\mu^{-1}(\alpha) P_\mu \eta(\alpha) P_\mu, \tag{7}$$

where the numbers $\gamma_\mu(\alpha)$ are the eigenvalues of the operator η from (4) given by

$$\gamma_\mu(\alpha) = N \frac{m_\mu d_\alpha}{m_\alpha d_\mu}, \tag{8}$$

where d_α, d_μ are dimensions of the irreps of $S(N-1), S(N)$ labelled by Young diagrams $\alpha \vdash N-1,\ \mu \vdash N$ respectively, and m_α, m_μ are their multiplicities.

By combining (7) and (8) we see that the PBT operator ρ is closely related to η and has the following form:

$$\rho = \sum_{\alpha - N-1} \sum_{\mu \in \alpha} \lambda_\mu(\alpha) F_\mu(\alpha), \tag{9}$$

where

$$\lambda_\mu(\alpha) = \frac{1}{d^N} \gamma_\mu(\alpha). \tag{10}$$

In our previous work [12] we give an explicit expression for the fidelity F given in equation (1) in terms of $N,\ d$, the dimensions d_μ, and multiplicities m_μ of irreps of the permutation group $S(N)$ when the resource state is given by as a N-fold tensor product of $|\psi^+\rangle$. In this case, we also know that optimal POVMs $\{\widetilde{\Pi}_a\}_{a=1}^N$ are given in the form of square root measurements (see (1)). In the qubit case when both the measurement and the resource state are optimised simultaneously it is known that it is possible to achieve a significantly higher teleportation fidelity [8]. In the latter case, the resource state differs from $|\psi^+\rangle^{\otimes N}$, and one has a different set of POVMs. In the qudit case, we similarly take the resource state to be

$$|\Psi\rangle = (O_A \otimes \mathbf{1}_B)|\psi^+\rangle_{A_1B_1} \otimes |\psi^+\rangle_{A_2B_2} \otimes \cdots \otimes |\psi^+\rangle_{A_NB_N}, \tag{11}$$

where $A = A_1A_2 \cdots A_N,\ B = B_1B_2 \cdots B_N$, and O_A encodes an arbitrary operator on Alice's side satisfying $\mathrm{Tr}\, O_A^\dagger O_A = d^N$. We want to compute

$$F = \frac{1}{d^2} \max_{\{\Pi_a\}} \sum_{a=1}^N \mathrm{Tr}[\Pi_a \sigma_a], \tag{12}$$

with respect to the following constraints

$$(1) \quad \sum_{a=1}^N \Pi_a \leqslant X_A \otimes \mathbf{1}_{\overline{B}}, \quad (2) \quad \mathrm{Tr}\, X_A = \mathrm{Tr}\, O_A^\dagger O_A = d^N, \tag{13}$$

where $\{\Pi_a\}_{a=1}^N$ is some new, optimal set of decorated POVMs[5] which are compatible with operation O_A and $\mathbf{1}_{\overline{B}}$ is identity operator acting on single qudit space on Bobs' side. We see that the problem of simultaneous optimisation over a resource state $|\Psi\rangle$ and the set of decorated POVMs $\{\Pi_a\}_{a=1}^N$ can be cast as a semidefinite programme (SDP) [3]. If we are interested in optimising only the measurement then see [12], and for explicit formula in the case of a small number of ports see [14]. Most of this work is dedicated to finding an optimal form of Alice' operation O_A, the optimal form of decorated POVMs, and the expression for the optimal value of the fidelity (12). As we have mentioned above we solve this problem by giving an analytical solution of the primal and the dual SDP. Moreover, all such solutions are presented in terms of objects characterising $\mathcal{A}_n^{t_n}(d)$.

[5] We use the phrase 'decorated POVMs' since they have been POVMs before applying operator O_A by Alice. This distinction is important since sum of Π_a is not smaller or equal to identity operator, so they cannot be called POVMs.

3. Facts about symmetric group $S(N)$

Before we state and prove our results, we need to introduce further group-theoretic notation.

(i) By the symbol $\nu/\mu = \square$ we denote two Young diagrams μ, ν for the same natural number N when μ can be obtained from ν by moving a single box $\square$ (and vice versa).

(ii) By $\alpha \in \mu$ we denote Young diagrams $\alpha \vdash N - 1$ which can be obtained from $\mu \vdash N$ by removing one box $\square$.

(iii) By $\widehat{S}(N)$ we denote the set of all possible irreps of the symmetric group $S(N)$, and by $|\widehat{S}(N)|$ its cardinality.

(iv) By $\varphi^{\alpha}, \psi^{\mu}$, etc we denote irreps of respective symmetric groups belonging to sets $\widehat{S}(N-1)$ or $\widehat{S}(N)$.

(v) For every permutation $\sigma \in S(N)$ we define its decomposition into disjoint cycles $\sigma \in (1^k, 2^{\xi_2}, \ldots, N^{\xi_N})$, where $k \geqslant 0, \xi_i \geqslant 0, i = 2, \ldots, N$ denote the number of cycles of length 1 to N. Moreover we have $1 \cdot k + 2\xi_2 + \ldots + N\xi_N = N$. For example for $\sigma = (23)(45)(678)$ we write $\sigma \in (1^1, 2^2, 3^1)$ and indeed we have $1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 = 8$.

Recall that the representations $\mathrm{Res}^{S(N)}_{S(N-1)}(\psi^{\nu}),\ \psi^{\nu} \in \widehat{S}(N)$ and $\mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})\ \varphi^{\alpha} \in \widehat{S}(N-1)$, have the following structure

$$\mathrm{Res}^{S(N)}_{S(N-1)}(\psi^{\nu}) = \bigoplus_{\alpha \in \nu} \varphi^{\alpha}, \qquad \mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha}) = \bigoplus_{\mu \in \alpha} \psi^{\mu}, \tag{14}$$

so they are simply reducible. The following properties of $\mathrm{Res}^{S(N)}_{S(N-1)}(\psi^{\nu})$ and $\mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})$ will be required in section 4:

Proposition 1. *We have the following:*

(a) $\varphi^{\alpha} \in \mathrm{Res}^{S(N)}_{S(N-1)}(\psi^{\nu})$ *if and only if* $\psi^{\nu} \in \mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})$.

(b) Irreps $\psi^{\mu}, \psi^{\nu} \in \widehat{S}(N)$, $\mu \neq \nu$ *are in the relation* $\nu/\mu = \square$ *if and only if there exists* $\varphi^{\alpha} \in \mathrm{Res}^{S(N)}_{S(N-1)}(\psi^{\nu}) : \psi^{\mu} \in \mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})$.

Proof. The statement (a) of the proposition is a well-known result in representation theory. We prove part (b). From the assumption we have

$$\nu = (\nu_1, \ldots, \nu_k, \ldots, \nu_l, \ldots, \nu_p) \Rightarrow \mu = (\nu_1, \ldots, \nu_k - 1, \ldots, \nu_l + 1, \ldots, \nu_p) \tag{15}$$

for some indices k, l. We chose

$$\alpha = (\nu_1, \ldots, \nu_k - 1, \ldots, \nu_l, \ldots, \nu_p) \vdash N - 1, \tag{16}$$

which is properly defined Young diagram because by assumption μ is properly defined Young diagram and we have $\mu \in \alpha$, so $\psi^{\mu} \in \mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})$. On the other hand for (b) we have from the assumption that for a given $\nu = (\nu_1, \ldots, \nu_s, \ldots, \nu_t, \ldots, \nu_q)$ such that $s \neq t$

$$\alpha = (\nu_1, \ldots, \nu_s, \ldots, \nu_t - 1, \ldots, \nu_p), \qquad \mu = (\alpha_1, \ldots, \alpha_s + 1, \ldots, \alpha_t, \ldots, \alpha_q), \tag{17}$$

so $\mu = (\nu_1, \ldots, \nu_s + 1, \ldots, \nu_t - 1, \ldots, \nu_q)$ and $\nu/\mu = \square$. $\square$

We further prove the following statement about characters of the induced representations.

Lemma 2. *Let* $\sigma \in S(N)$ *and suppose that σ has the following cycle structure* $\sigma \in (1^k, 2^{\xi_2}, \ldots, N^{\xi_N})$, *then*

$$\chi^{\mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})}(\sigma) = k\chi^{\alpha}(1^{k-1}, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}). \tag{18}$$

In particular for $\sigma = e \in (1^N)$, *where e denotes identity element of the group* $S(N)$ *we have*

$$\chi^{\mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha})}(e) = Nd_{\alpha}. \tag{19}$$

Proof. Recall that the induced representation $\mathrm{Ind}^{S(N)}_{S(N-1)}(\varphi^{\alpha}) : \varphi^{\alpha} \in \widehat{S}(N-1)$ has the following form

$$\forall\, \sigma \in S(N) \qquad \Phi^{\mathrm{Ind}(\varphi^{\alpha})}_{ai,bj}(\sigma) = \widetilde{\varphi}^{\alpha}_{ij}[(aN)\sigma(bN)], \tag{20}$$

where

$$\widetilde{\varphi}^{\alpha}_{ij}(\pi) = \begin{cases} \varphi^{\alpha}(\pi),\ \pi \in S(N-1), \\ 0,\ \pi \notin S(N-1), \end{cases} \tag{21}$$

and $a, b = 1, \ldots, N$. We thus get the following formula for the character of the induced representation

$$\chi^{\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})}(\sigma) = \sum_{i=1}^{d_{\alpha}} \sum_{a=1}^{N} \widetilde{\varphi}_{ii}^{\alpha}[(aN)\sigma(aN)] = \sum_{a=1}^{N} \widetilde{\chi}^{\alpha}[(aN)\sigma(aN)], \tag{22}$$

where $\widetilde{\chi}^{\alpha}$ is defined in the same way as $\widetilde{\varphi}_{ij}^{\alpha}$. Let $\sigma = C_1 C_2 \cdots C_k \in S(N)$ be a unique decomposition of the permutation σ into disjoint cycles. For a given transposition (aN) of the natural transversal, the number a appears in only one cycle C_i in σ, and similarly for the number N and we have the following possible cycles, which include the numbers a, N

$$\begin{aligned} &(aN)(ai_1 \cdots i_p)(aN) = (Ni_1 \cdots i_p), \quad i_k \neq N, \\ &(aN)(Ni_1 \cdots i_p)(aN) = (ai_1 \cdots i_p), \quad i_k \neq a, \\ &(aN)(ai_1 \cdots N \cdots i_p)(aN) = (Ni_1 \cdots a \cdots i_p), \quad i_k \neq a, N. \end{aligned} \tag{23}$$

From equation (23) it follows that if $\sigma = C_1 C_2 \cdots C_k \in S(N)$ is such that $|C_i| > 1$ (i.e. all cycles C_i in σ are of the length greater than one), then for any transposition (aN) the permutation $(aN)\sigma(aN)$ does not belong to $S(N-1)$, and $\chi^{\mathrm{Ind}_{S(N-1)}^{S(N)}}(\sigma) = \sum_{a=1}^{N} \widetilde{\chi}^{\alpha}[(aN)\sigma(aN)] = 0$. Suppose now that a permutation σ contains the cycle of the length one i.e. it is of the form

$$\sigma \in (1^k, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}), \quad k \geqslant 1, \ \xi_j \geqslant 0, \qquad \sigma = (a_1)(a_2) \cdots (a_k) C_1 \cdots C_p, \tag{24}$$

where $a_i = 1, \ldots, N$ and $|C_j| > 1$. In this case, we have for $i = 1, \ldots, k$

$$(a_i N)\sigma(a_i N) = (a_1) \cdots (N) \cdots (a_k) C_1{}' \cdots C_p{}' \in S(N-1), \tag{25}$$

so for k transpositions of the transversal $(a_i N)$: $i = 1, \ldots, k$ we have

$$\widetilde{\chi}^{\alpha}[(a_i N)\sigma(a_i N)] = \chi^{\alpha}(1^{k-1}, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}) \tag{26}$$

and for the remaining transpositions of the transversal $(a_j N) : j > k$ we have

$$\widetilde{\chi}^{\alpha}[(a_j N)\sigma(a_j N)] = 0, \tag{27}$$

and

$$\chi^{\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})}(1^k, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}) = k\chi^{\alpha}(1^{k-1}, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}). \tag{28}$$

□

4. Teleportation matrix

We are now ready to define the central object of our work—the TM M_F which plays a key role in the analysis of the simultaneous optimisation over decorated POVMs and the resource state in the dPBT. Later, we will derive a connection between M_F and induced characters of the symmetric group which enables us to use results from section 3 in order to determine its spectral properties. We provide an analytical expression for its eigenvalues whenever $d \geqslant N$, and show that M_F together with all of its principal submatrices is positive semidefinite. Finally, we derive a few other important properties of M_F like its irreducibility and primitivity which are necessary when we discuss the convergent algorithm for computation of the infinity norm of principal submatrices of M_F (i.e. when $d < N$ and the closed-form analytical expression for the eigenvalues is not known).

Definition 3. Let μ, ν run over all irreps of the group $S(N)$, define the following matrix M_F of dimension $|\widehat{S}(N)|$

$$M_F \equiv (n_{\mu}\delta_{\mu,\nu} + \Delta_{\mu,\nu}), \tag{29}$$

where n_{μ} is the number of $\alpha \vdash N-1$ for which $\alpha \in \mu$, and

$$\Delta_{\mu,\nu} = \begin{cases} 1 \text{ if } \mu/\nu = \square, \\ 0 \text{ otherwise.} \end{cases} \tag{30}$$

The symbol $\mu/\nu = \square$ denotes such Young diagrams μ, ν which can be obtained from each other by moving a single box.

Figure 1 depicts M_F for $N = 2, 3, 4$ when all the irreps of $S(N)$ occur. From the representation theory point of view, the structure of M_F encodes relations among the irreps of the group $S(N)$. As we will see later, the relations that define the matrix M_F are determined by the properties of the representations Res and Ind (see section 3). We will further assume that all indices $\psi^{\mu}, \psi^{\nu} \in \widehat{S}(N)$ of the matrix M_F are ordered in the strongly decreasing lexicographic order, starting from the biggest Young diagram $\mu = (N)$. In such ordering, Young diagrams strongly decrease, whereas the height of the Young diagrams weakly increases.

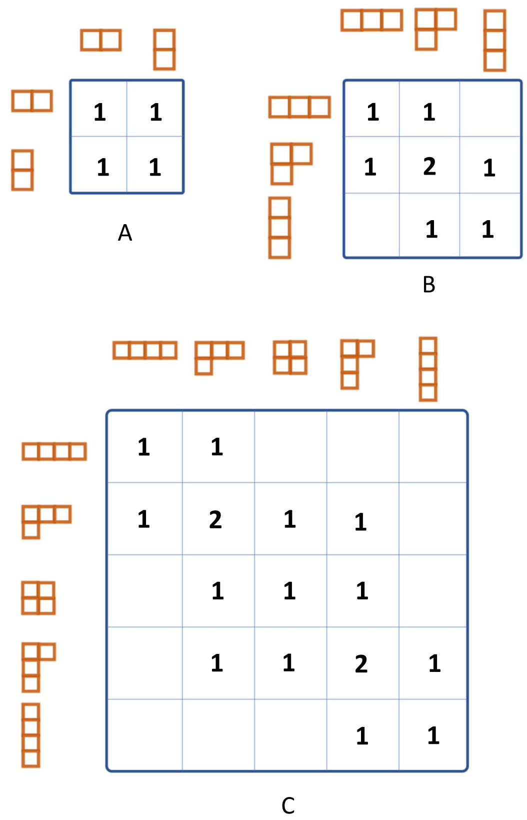

Figure 1. Teleportation matrix for the dPBT schemes. The maximal eigenvalue of each of the matrices determines the optimal performance of the dPBT scheme for $N = 2$ (A), $N = 3$ (B), $N = 4$ (C) in the case where all the irreps occur (i.e. local dimension d of the teleported state and each of the port equals to N). Empty squares are filled with zeros.

To reveal the connection between M_F and irreps of $S(N)$ we start from the following lemma:

Lemma 4. *The numbers, which appear in the row ν of the matrix M_F, are the multiplicities of the irreps $\psi^\nu \in \widehat{S}(N)$ appearing in all representations*

$$\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha) : \varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu), \qquad \varphi^\alpha \in \widehat{S}(N-1), \tag{31}$$

where the diagonal term n_ν shows how many $\varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$.

Proof. The lemma is in fact, a corollary from the proposition 1. From the statement (a) of this proposition we get that for a given $\psi^\nu \in \widehat{S}(N)$, so for a given row ν of the matrix M_F, the irrep ν is included in all representations $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha)$ such that $\varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$, and there are n_ν of them. From statement (b) of proposition 1 we get that if $\mu \neq \nu$ then $\nu/\mu = \Box$ if and only if ψ^μ belongs to $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha)$ for some $\varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$. It is not difficult to prove that in the case $\mu \neq \nu$ the irrep $\mu : \nu/\mu = \Box$ appears only once in all $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha) : \varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$. □

From the point of view of representation theory, the structure of M_F encodes relations among the irreps of $S(N)$. Such relations are determined by the properties of the representations Res and Ind (see section 3). In what follows we assume that all indices ψ^μ, $\psi^\nu \in \widehat{S}(N)$ of the matrix M_F are in the strongly decreasing, lexicographic order, starting from $\mu = (N)$. In such ordering Young diagrams strongly decrease, whereas their heights weakly increase.

To reveal the connection between M_F and irreps of $S(N)$ the first prove the following lemma:

Lemma 5. *The numbers, which appear in the row ν of the matrix M_F, are the multiplicities of the irreps $\psi^\nu \in \widehat{S}(N)$ appearing in all representations*

$$\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha) : \varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu), \qquad \varphi^\alpha \in \widehat{S}(N-1), \tag{32}$$

where the diagonal term n_ν shows how many $\varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$.

Proof. The lemma is in fact, a corollary from the proposition 1. From the statement (a) of this proposition we get that for a given $\psi^{\nu} \in \widehat{S}(N)$, so for a given row ν of the matrix M_F, the irrep ψ^{ν} is included in all representations $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})$ such that $\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})$, and there are n_{ν} of them. From statement (b) of proposition 1 we get that if $\mu \neq \nu$ then $\nu/\mu = \square$ if and only if ψ^{μ} belongs to $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})$ for some $\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})$. It is not difficult to prove that in the case $\mu \neq \nu$ the irrep $\mu : \nu/\mu = \square$ appears only once in all $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha}) : \varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})$. □

In order to describe the spectral properties of the matrix M_F we introduce a notion of reduced character

Definition 6. The reduced character matrix for the group $S(N)$ has the following form

$$T \equiv (\chi_{\mu}(C)), \tag{33}$$

where μ runs over all irreps of the group $S(N)$, $C = (1^k, 2^{\xi_2}, \ldots, N^{\xi_N})$ describes the class of conjugated elements, $\chi_{\mu}(\cdot)$ is character calculated on irrep μ and elements from C. By $T(C) = (\chi_{\mu}(C))$, where C runs over all classes of the group $S(N)$, we denote the columns of the matrix T.

Matrix $T = (\chi_{\mu}(C))$ is unitary and related to M_F via:

Proposition 7. *We have the following spectral properties of the matrix M_F*

$$M_F T(C) = kT(C) \Leftrightarrow \sum_{\mu} (M_F)_{\nu\mu} \chi_{\mu}(C) = k\chi_{\nu}(C), \tag{34}$$

where $C = (1^k, 2^{\xi_2}, \ldots, N^{\xi_N})$, so k is the number of cycles of the length 1 in the class C which is the support of the eigenvector $T(C)$. The reduced character matrix T for the group $S(N)$, diagonalises the matrix M_F.

Proof. From lemma 5 we deduce that for the given row ν of the matrix M_F the sum

$$\sum_{\mu} (M_F)_{\nu\mu} \chi_{\mu}(C) \tag{35}$$

is equal to the sum of all characters of the irreps of the group $S(N)$ which are included in all induced representations $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha}) : \varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})$, $\varphi^{\alpha} \in \widehat{S}(N-1)$ i.e. we have

$$\sum_{\mu} (M_F)_{\nu\mu} \chi_{\mu}(C) = \sum_{\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})} \chi^{\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})}(C), \tag{36}$$

where $C = (1^k, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}})$. From lemma 2 we have

$$\sum_{\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})} \chi^{\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^{\alpha})}(C) = k \sum_{\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})} \chi^{\alpha}(1^{k-1}, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}), \tag{37}$$

where the sum on *RHS* is the character of the representation $\mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})$, and we have

$$\sum_{\varphi^{\alpha} \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^{\nu})} \chi^{\alpha}(1^{k-1}, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}) = \chi_{\nu}(1^k, 2^{\xi_2}, \ldots, (N-k)^{\xi_{N-k}}) = \chi_{\nu}(C). \tag{38}$$

□

From proposition 7 one can get:

Corollary 8.

1. *The matrix M_F has the following spectrum*

$$\mathrm{spec}(M_F) = \{0, 1, 2, \ldots, N-2, N\}. \tag{39}$$

Note that there is a gap in this spectrum—the number $N-1$ does not occur.

2. *The matrix M_F is positive semidefinite.*

3. *The multiplicity of the eigenvalue $k \in \mathrm{spec}(M_F)$ is equal to the number of cycles classes of the form $(1^k, 2^{\xi_2}, \ldots, N^{\xi_N})$, equivalently to the number of solutions in $\mathbb{N} \cup \{0\}$ of the equation (equations for ξ_l)*

$$\sum_{l=2}^{N-k} l\xi_l = N - k. \tag{40}$$

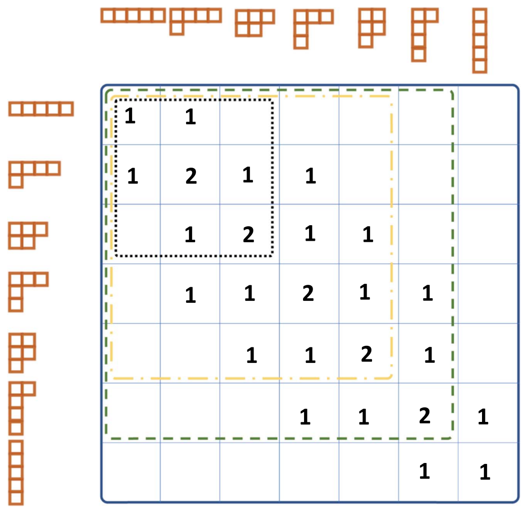

Figure 2. Teleportation matrix for the dPBT schemes with fixed number of ports ($N = 5$—number of boxes in each shape) and varying dimensions of each port and teleported state (the maximum admissible height of each shape). A sequence of principal submatrices corresponds to an optimal performance of a different dPBT scheme: the entire matrix (solid blue frame) corresponds to $d \geqslant 5$, and its first principal submatrix (dashed green frame) corresponds to the dPBT $d = 4$, followed by $d = 3,2$ (dashed–dotted yellow frame and dotted black frame respectively). Empty cells contain zeros.

4. *The eigenvector $v = (v_\mu)$ for $\mu \in \widehat{S}(N)$ corresponding to maximal eigenvalue N has strictly positive entries (which agrees with Frobenius–Perron theorem—see theorem 44 of appendix B) and $\forall\, \mu \in \widehat{S}(N)$ $v_\mu = d_\mu$, where d_μ is the dimension of the respective irrep.*
5. *The largest eigenvalue N, in fact spectral radius, has multiplicity one, which agrees with Frobenius–Perron Theorem. Similarly the eigenvalues $N-2, N-3$ also are simple and the multiplicities of the eigenvalues $N-4, N-5$ are equal 2 and so on.*

The above statements are true when the dimension d of the underlying Hilbert space is large enough, i.e. whenever heights $h(\mu), h(\nu)$ of Young diagrams labelling rows and columns of M_F satisfy conditions $h(\mu) \leqslant d, h(\nu) \leqslant d$, otherwise some irreps do not exist. The minimal dimension d for having all irreps is just equal to the height of the Young diagram corresponding to antisymmetric space, so it occurs when $d \geqslant N$.

To make our exposition more transparent, we introduce the following

Definition 9. If $\psi^\mu \in \widehat{S}(N)$ is irrep of the group $S(N)$ we write

$$\widehat{S}_d(N) = \{\psi^\mu \in \widehat{S}(N) : h(\mu) \leqslant d\} \Rightarrow \widehat{S}_N(N) = \widehat{S}(N). \tag{41}$$

Thus whenever d is small that the height of a for Young diagrams spectral analysis reduces to that of the respective principal submatrices of M_F defined as follows

Definition 10. By M_F^d we denote a principal submatrix (i.e. matrix localised on the main diagonal in the upper left corner), which contains all irreps $\psi^\nu \in \widehat{S}(N)$, such that $h(\nu) \leqslant d$. For such choice we have

$$N \leqslant d \Rightarrow M_F^d = M_F, \tag{42}$$

and in particular $M_F^N = M_F$.

Figure 2 illustrates M_F with its principal submatrices M_F^d for $N = 5$ when $d = 2, 3, 4, 5$.

Remark 11. From Sylvester's theorem (see theorem 41 of appendix B) it follows that all principal matrices M_F^d are positive semidefinite.

Using lemma 5 we can calculate how many irreps ψ^ν of $S(N)$ we have in $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha)$: $\varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$ (i.e. how many $1's$ (with multiplicities) we have in the row ν in the matrix M_F^d):

Proposition 12. *The number of all $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha) : \varphi^\alpha \in \mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\nu)$ is not greater than $h(\nu) \leqslant d$, so $n_\nu \leqslant d$. In each induced representation $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha)$ we have at most $h(\nu) + 1$ irreps of $S(N)$, if $h(\nu) < d$, and d irreps of*

$S(N)$ if $h(\nu) = d$. From this it follows that in the matrix M_F^d, the maximum number of 1's (with multiplicities) in each row is not greater than d^2.

Defining $||A|| \equiv \max_i \sum_{j=1} |a_{ij}|$, for an arbitrary $A = (a_{ij}) \in \mathbb{M}(n, \mathbb{C})$, and using proposition 12 we have the following

Corollary 13. *We have the following upper bound for the norm of the matrix norm of M_F^d*

$$||M_F^d|| \leqslant d^2. \tag{43}$$

We now prove a few additional important features of the TM M_F, and its principal matrices M_F^d. It turns out that matrices M_F^d have a few useful properties regarding our algorithm presented further in section 5.4—irreducibility and primitivity which are explained in definitions 42, 43, and 45 of appendix B.

Fact 14. M_F given in definition 3 is irreducible in the sense of definition 43.

Proof. The formal proof can be done by induction, but the following observation is enough. From the definition 3 we see that the matrix M_F is at least three-diagonal. The number of zeros in every row of the matrix M_F is equal then to $m = |\widehat{S}(N)| - 2$. After the exponentiation of M_F^2 the positions $(M_F)_{1,3} \neq 0, \ldots, (M_F)_{|\widehat{S}(N)|-2, |\widehat{S}(N)|} \neq 0$, so the third upper (lower) diagonal becomes non-zero. Computing M_F^3 we see that the fourth upper (lower) diagonal has strictly positive entries. Because of the construction continuing process of the multiplication $m + 1 = |\widehat{S}(N)| - 1$ times we have $(A^{m+1})_{ij} > 0$ for every $1 \leqslant i, j \leqslant |\widehat{S}(N)|$. In general matrix M_F has strictly positive numbers also outside of the three main diagonals. It means that in the general case the required number of the multiplications can be smaller than $m + 1$. □

Using similar arguments as in fact 14 we can show that every principal matrix M_F^d is also irreducible. Matrix M_F, and its principal matrices M_F^d are also primitive matrices (see definition 45 of appendix B). Matrices M_F, M_F^d satisfy all the assumptions of proposition 46 of appendix B so we get:

Corollary 15. *The matrices M_F, M_F^d are primitive.*

Remark 16. It follows also directly from the positive semidefiniteness of the matrices M_F^d.

And lastly

Remark 17. The matrix M_F given in the definition 3 is a centrosymmetric matrix according to definition 47 of appendix B.

4.1. A different approach to eigenvalue analysis of the TM

We will now exhibit an entirely different approach to finding spectrum of M_F. Recall definition 9 and define the following matrix:

Definition 18. For every $N \geqslant 2$ we define

$$R_N^d \equiv (r_{\alpha\mu}^d(N)) \in \mathbb{M}(\widehat{S}_d(N-1) \times \widehat{S}_d(N), \mathbb{Z}), \tag{44}$$

where

$$r_{\alpha\mu}^d(N) = \begin{cases} 1 : \mu \in \alpha, \\ 0 : \mu \notin \alpha. \end{cases} \tag{45}$$

The matrix R_N^d has its rows indexed by irreps $\varphi^\alpha \in \widehat{S}_d(N-1)$ whereas the columns are indexed by irreps $\psi^\mu \in \widehat{S}_d(N)$. The irreps indices of the matrix R_N^d are ordered lexicographically and we set $R_N^N = R_N$.

The matrix R_N^d has the following interesting properties:

(1) The sum of 1's in a given row α is equal to the number of irreps $\psi^\mu \in \widehat{S}_d(N)$ included in the representation $\mathrm{Ind}_{S(N-1)}^{S(N)}(\varphi^\alpha)$.

(2) The sum of 1's in a given column μ is equal to the number of irreps $\varphi^\alpha \in \widehat{S}_d(N-1)$ included in the representation $\mathrm{Res}_{S(N-1)}^{S(N)}(\psi^\mu)$.

(3) The number 1 in the position (α, μ) in R_N^d means that the projector $F_\mu(\alpha)$ is non-zero.

Example 19. In this example we show the explicit form of matrix R_N^d given in definition 18 for $d = N = 4$:

$$R_4^4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{46}$$

Matrices R_N^d have the following property:

Proposition 20. *For any $d \geqslant 2$ and $N \geqslant 2$ the matrix R_N^d has maximal rank equal $|\widehat{S}_d(N-1)|$, so the rows of the matrix R_N^d are linearly independent.*

Proof. Let consider a square submatrix of maximal dimension whose columns are indexed by irreps $\psi^\mu \in \widehat{S}_d(N)$

$$\mu = \alpha + \square, \tag{47}$$

where the box is added to the first row of α which labels $\varphi^\alpha \in \widehat{S}_d(N-1)$, so Young diagrams μ are ordered similarly to α. Then one can show that such a square matrix is upper triangular with $1's$ on the diagonal, therefore the corresponding minor of maximal dimension is non-zero. $\square$

We now define two other matrices which are connected with R_N^d:

Definition 21.

$$G_N^d \equiv (g_{\mu\nu}^d(N)) = (R_N^d)^T R_N^d \in \mathbb{M}(\widehat{S}_d(N), \mathbb{Z}), \tag{48}$$

$$H_N^d \equiv (h_{\alpha\beta}^d(N)) = R_N^d (R_N^d)^T \in \mathbb{M}(\widehat{S}_d(N-1), \mathbb{Z}), \tag{49}$$

each of which is Gram matrix of the columns of the matrix R_N^d and Gram matrix of the rows the matrix R_N^d respectively. The matrix G_N^d is indexed by Young diagrams μ such that $\psi^\mu \in \widehat{S}_d(N)$ whereas the matrix H_N^d is indexed by Young diagrams α such that $\varphi^\alpha \in \widehat{S}_d(N-1)$.

From proposition 20 it follows that the matrix H_N^d is invertible with the following connection between the spectra of the matrices G_N^d and H_N^d:

Proposition 22. *All non-zero eigenvalues of the matrix G_N^d are precisely the eigenvalues of the matrix H_N^d and the corresponding eigenvectors are related by matrix R_N^d. In particular, matrices G_N^d and H_N^d have the same spectral radius.*

We now show that matrices R_N^d, G_N^d, and H_N^d are closely related to principal submatrices $M_F^d(N)$ of TM M_F given in definitions 10 and 3 respectively:

Theorem 23. *The following relation holds*

$$G_N^d = M_F^d(N), \tag{50}$$

so the matrix $M_F^d(N)$ is in fact a Gram matrix.

Proof. Let consider the matrix element of the matrix G_N^d (we omit here the index N)

$$g_{\mu\nu}^d = \sum_\alpha r_{\mu\alpha}^d r_{\alpha\nu}^d. \tag{51}$$

If $\mu = \nu$, then the non-zero terms in the sum on RHS of (51) are those for $\alpha = \mu - \square$, so $h(\alpha) \leqslant h(\mu)$ and the summation of $1's$ is over those α labelling $\varphi^\alpha \in \widehat{S}_d(N-1)$, from which one obtains μ by adding properly one box to and $\psi^\mu \in \widehat{S}_d(N)$. Therefore $g_{\mu\mu}^d = (M_F^d)_{\mu\mu}$.

If $\mu \neq \nu$, then the non-zero terms in the sum on RHS of (51) are for such α labelling $\varphi^\alpha \in \widehat{S}_d(N-1)$, for which one obtains both μ, ν by adding one box to α and $\psi^\mu, \psi^\nu \in \widehat{S}_d(N)$. There exists only one such Young diagram α and it means that the Young diagrams μ, ν are such that one is obtained from another one by moving one box, which is a definition of the element $(M_F^d)_{\mu\nu}$ in the matrix $M_F^d(N)$. $\square$

Corollary 24. *For any $d \geqslant 2$ and $N \geqslant 2$ the matrix $M_F^d(N)$ is positive semidefinite.*

Proving semidefiniteness of $M_F^d(N)$ becomes straightforward when we adopt the approach of this subsection. To derive the remaining result we need the following simple observation:

Remark 25. For any $d \geqslant 2$ and $N \geqslant 2$ the matrix R_N^d is a principal submatrix of the full matrix R_N.

As well as two technical lemmas below:

Lemma 26. *Fix two irreps $\varphi^\alpha \in \widehat{S}(N-1)$ and $\psi^\mu \in \widehat{S}(N)$. If a Young diagram α is such that $\alpha = \mu - \Box$ i.e.*

$$\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_k) = (\mu_1, \ldots, \mu_i - 1, \ldots, \mu_k), \tag{52}$$

then $\gamma = (\mu_1, \ldots, \mu_{i-1} - 1, \mu_i - 1, \ldots, \mu_k) \vdash N - 2$ is also a well defined Young diagram and it labels an irrep of $S(N-2)$.

Lemma 27. *Consider the matrix R_N^d as a principal submatrix of the full matrix R_N, then the row labelled by $\alpha : h(\alpha) < d$ of the submatrix R_N^d includes all 1's from the row labelled by α in the matrix R_N. If the row labelled by α of the submatrix R_N^d is such that $h(\alpha) = d$, then there is a single 1, which is outside the submatrix R_N^d.*

Using these statements one can prove the following important relation between the matrices $M_F^d(N-1)$ and H_N^d

Theorem 28. *For any $d \geqslant 2$ and $N \geqslant 2$ we have*

$$H_N^d = J_p + M_F^d(N-1), \tag{53}$$

where the matrix J_p is of the form

$$J_p = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & 0 \end{pmatrix} \tag{54}$$

and $\mathbf{1}_p$ is the identity matrix of dimension p, which is the number of rows α for which $\varphi^\alpha \in \widehat{S}_d(N-1)$ of the submatrix R_N^d is such that $h(\alpha) = d$.

In particular we have

$$H_N^2 = J_1 + M_F^2(N-1), \qquad H_N^N = \mathbf{1} + M_F(N-1), \tag{55}$$

i.e. in the last case J_p is a identity matrix.

Remark 29. The importance of theorem 28 follows from the fact that the matrices H_N^d and $G_N^d = M_F^d(N)$ have the same non-zero eigenvalues (see proposition 22), so the relation in the theorem yields a recursive formula between eigenvalues, matrices $M_F^d(N)$ and $M_F^d(N-1)$.

The starting point of the recursive descent is the case $d = N$ which then gives a following recursive relation for the maximal eigenvalues $\lambda_{\max}(N)$ of matrices $M_F(N)$

$$\lambda_{\max}(N) = 1 + \lambda_{\max}(N-1) \Rightarrow \lambda_{\max}(N) = N, \tag{56}$$

which coincides with the earlier result obtained using the spectral decomposition of the matrix $M_F(N)$ but with significantly less effort.

5. Optimisation over a resource state in the dPBT

We now turn to the case when both the resource state $|\Psi\rangle$ and Alice's decorated POVMs $\{\Pi_a\}_{a=1}^N$ are optimised simultaneously. Since from [8] we know that this problem can be cast in terms of SDP, we provide analytical solutions to both primal and dual SDPs and find the optimal form of decorated POVMs and the state $|\Psi\rangle$.

From general theory of SDP the solution of the primal problem provides only an upper bound on the actual solution. On the other hand, we know that the solution of the dual problem lower bounds it. In our case it turns out that the solution of the primal problem equals to that of a dual (which is not always the case for an arbitrary SDP), so by showing that the primal matches the dual, we obtain the optimal fidelity. The optimal fidelity of the dPBT is directly expressed in terms of the TM M_F given in definition 3 or its principal matrices if the dimension d is smaller than the number of the ports N. More precisely, it is given by the square of a maximal eigenvalue divided by the square of the dimension of the teleported system.

Figure 3 illustrates how optimal fidelity compares to previous results.

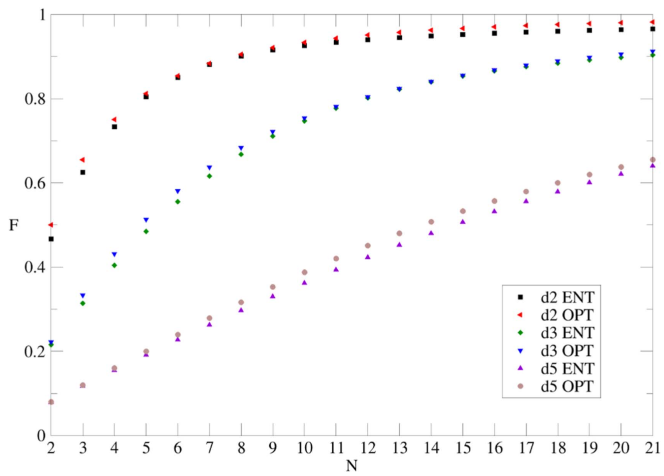

Figure 3. Best achievable fidelity of port-based teleportation when both the state and the measurement is optimised. dX ENT denotes the fidelity of the dPBT when the resource state consists of maximally entangled pairs and only the measurement is optimised; X corresponds to the dimension of the teleported state. dX OPT denotes the best possible fidelity achieved by optimizing the resource state and measurement simultaneously.

5.1. The primal SDP problem

The primal problem is to compute:

$$F^* = \frac{1}{d^2} \max_{\{\Pi_i\}} \sum_{a=1}^{N} \mathrm{Tr}[\Pi_a \sigma_a], \tag{57}$$

with respect to constraints

$$(1) \quad \sum_{a=1}^{N} \Pi_a \leqslant X_A \otimes \mathbf{1}_{\overline{B}}, \quad (2) \quad \mathrm{Tr}\, X_A = d^N. \tag{58}$$

In the above $\{\Pi_a\}_{a=1}^N$ is the set of decorated POVMs used by Alice, and $X_A = O_A^\dagger O_A$, where O_A is a global operation performed on Alices' half of the maximally entangled resource state. The solution of (57) with the constraints (58) is given in the following

Theorem 30. *The quantity F^* in the primal problem can be expressed as:*

$$F^* = \frac{1}{d^2} ||M_F^d||_\infty, \tag{59}$$

where $||M_F^d||_\infty$ denotes the infinity norm of the principal submatrix of the TM M_F which are given in definitions 10 and 3 respectively, and d *denotes the dimension of the port.*

Proof. Here we assume the most general form of the decorated POVMs (indeed more general than in (110)); for $a = 1, \ldots, N$ we take:

$$\Pi_a = \Pi \sigma_a \Pi, \tag{60}$$

with

$$\Pi = \sum_\alpha \sum_{\mu \in \alpha} p_\mu(\alpha) F_\mu(\alpha), \quad p_\mu(\alpha) \geqslant 0, \tag{61}$$

and

$$X_A = \sum_\mu c_\mu P_\mu, \quad c_\mu \geqslant 0. \tag{62}$$

We rewrite expression (57) using our assumption about the form of decorated POVMs Π_a for $a = 1, \ldots, N$ given in (60):

$$\begin{aligned} F^* &= \frac{1}{d^2} \max_{\{\Pi_a\}} \mathrm{Tr}\left[\sum_{a=1}^{N} \Pi_a \sigma_a\right] = \frac{1}{d^2} \max_{\Pi} \sum_a \mathrm{Tr}[\Pi \sigma_a \Pi \sigma_a] \\ &= \frac{N}{d^2} \max_{\Pi} \mathrm{Tr}[\Pi \sigma_N \Pi \sigma_N] = \frac{N}{d^{2N}} \max_{\Pi} \mathrm{Tr}[\Pi(\mathbf{1} \otimes P_+)\Pi(\mathbf{1} \otimes P_+)], \end{aligned} \tag{63}$$

where we use the fact that $\mathrm{Tr}[\Pi_a \sigma_a]$ does not depend on the index $a = 1, \ldots, N$. This property allows us to compute the trace for fixed value $a = N$ and multiply it N times. Here and further in this manuscript by P_+ we denote projector onto the maximally entangled state $|\Phi^+\rangle$ between Nth and nth subsystem, and the identity operator $\mathbf{1}$ on $N-1$ first subsystems. Substituting decomposition of Π given in (61), fact that $\mathbf{1}\otimes P_+ = \frac{1}{d}V^{t_n}(N,n)$, and decomposition (5) we write:

$$F^* = \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha')\}} \sum_{\alpha,\alpha'} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha'}} p_\mu(\alpha) p_{\mu'}(\alpha') \mathrm{Tr}[M_\alpha P_\mu V^{t_n}(N,n) M_{\alpha'} P_{\mu'} V^{t_n}(N,n)]. \tag{64}$$

Using that $V^{t_n}(N,n)M_\alpha = V^{t_n}(N,n)P_\alpha$ (see Fact 13 of [12]) we have

$$F^* = \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha')\}} \sum_{\alpha,\alpha'} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha'}} p_\mu(\alpha) p_{\mu'}(\alpha') \mathrm{Tr}[P_\mu V^{t_n}(N,n) P_{\alpha'} P_{\mu'} V^{t_n}(N,n) P_\alpha]. \tag{65}$$

Using properties $[P_\alpha, V^{t_n}(N,n)] = 0$, $[P_\alpha, P_\mu] = 0$, $P_\alpha P_{\alpha'} = \delta_{\alpha\alpha'}P_\alpha$, and again $V^{t_n}(N,n)M_\alpha = V^{t_n}(N,n)P_\alpha$ we reduce above expression to

$$\begin{aligned} F^* &= \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_\mu(\alpha) p_{\mu'}(\alpha) \mathrm{Tr}[P_\mu V^{t_n}(N,n) P_\alpha P_{\mu'} V^{t_n}(N,n) P_\alpha] \\ &= \frac{N}{d^{2N}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_\mu(\alpha) p_{\mu'}(\alpha) \mathrm{Tr}[F_\mu(\alpha)(P_\alpha\otimes P_+) F_{\mu'}(\alpha)(P_\alpha\otimes P_+)]. \end{aligned} \tag{66}$$

In the next step we use of the identity operator in the form $\mathbf{1} = \sum_\alpha P_\alpha = \sum_\alpha \sum_{k=1}^{d_\alpha}\sum_{r=1}^{m_\alpha} |\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)|$, where vectors $\{|\varphi_{k,r}(\alpha)\rangle\}_{k=1}^{d_\alpha}$ span rth block of the irrep labelled by Young diagram α:

$$\begin{aligned} F^* = &\frac{N}{d^{2N}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_{\mu'}(\alpha) p_\mu(\alpha) \\ &\times \sum_{k,l=1}^{d_\alpha} \sum_{r,s=1}^{m_\alpha} \mathrm{Tr}[F_\mu(\alpha)|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)| \otimes P_+ F_{\mu'}(\alpha)|\varphi_{l,s}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)| \otimes P_+] \\ = &\frac{N}{d^{2N}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_{\mu'}(\alpha) p_\mu(\alpha) \\ &\times \sum_{k,l=1}^{d_\alpha} \sum_{r,s=1}^{m_\alpha} \mathrm{Tr}[|\varphi_{l,s}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)| \otimes P_+ F_{\mu'}(\alpha)] \mathrm{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)| \otimes P_+ F_\mu(\alpha)]. \end{aligned} \tag{67}$$

In the last step we made use of the following observation:

$$\begin{aligned} &\mathrm{Tr}[F_\mu(\alpha)|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)| \otimes P_+ F_{\mu'}(\alpha)|\varphi_{l,s}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)| \otimes P_+] \\ &= \mathrm{Tr}[F_\mu(\alpha)|\varphi_{k,r}(\alpha)\rangle \otimes |\Phi^+\rangle\langle\varphi_{k,r}(\alpha)| \otimes \langle\Phi^+|F_{\mu'}(\alpha)|\varphi_{l,s}(\alpha)\rangle \otimes |\Phi^+\rangle\langle\varphi_{l,s}(\alpha)| \otimes \langle\Phi^+|] \\ &= \langle\varphi_{k,r}(\alpha)| \otimes \langle\Phi^+|F_{\mu'}(\alpha)|\varphi_{l,s}(\alpha)\rangle \otimes |\Phi^+\rangle \mathrm{Tr}[F_\mu(\alpha)|\varphi_{k,r}(\alpha)\rangle \otimes |\Phi^+\rangle\langle\varphi_{l,s}(\alpha)| \otimes \langle\Phi^+|] \\ &= \mathrm{Tr}[|\varphi_{l,s}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)| \otimes P_+ F_{\mu'}(\alpha)] \mathrm{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)| \otimes P_+ F_\mu(\alpha)]. \end{aligned} \tag{68}$$

Using fact 39 we can simplify above expression as

$$\begin{aligned} F^* &= \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_{\mu'}(\alpha) p_\mu(\alpha) \frac{m_{\mu'} m_\mu}{m_\alpha^2} \sum_{k,l=1}^{d_\alpha} \sum_{r,s=1}^{m_\alpha} \delta_{lk}^2 \delta_{sr}^2 \\ &= \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha), p_{\mu'}(\alpha)\}} \sum_{\alpha} \frac{d_\alpha}{m_\alpha} \sum_{\substack{\mu\in\alpha\\ \mu'\in\alpha}} p_{\mu'}(\alpha) p_\mu(\alpha) m_{\mu'} m_\mu \\ &= \frac{N}{d^{2N+2}} \max_{\{p_\mu(\alpha)\}} \sum_{\alpha} \frac{d_\alpha}{m_\alpha} \left(\sum_{\mu\in\alpha} p_\mu(\alpha) m_\mu \right)^2. \end{aligned} \tag{69}$$

Form the definition of Π we see that $\forall\, \pi \in S(N)$ $[\Pi, V(\pi)] = 0$. Together with (9) we write

$$\sum_{a=1}^{N} \Pi_a = \Pi \sum_{a=1}^{N} \sigma_a \Pi = \Pi\rho\Pi = \Pi^2\rho = \sum_{\alpha}\sum_{\mu\in\alpha} p_\mu^2(\alpha)\lambda_\mu(\alpha) F_\mu(\alpha). \tag{70}$$

Similarly to equation (37) in [8] we get

$$\sum_{a=1}^{N} \Pi_a = \sum_{\alpha}\sum_{\mu\in\alpha} p_\mu^2(\alpha)\lambda_\mu(\alpha)F_\mu(\alpha) = \sum_{\mu}\sum_{\alpha\in\mu} p_\mu^2(\alpha)\lambda_\mu(\alpha)F_\mu(\alpha) \leqslant \sum_{\mu} c_\mu P_\mu \otimes \mathbf{1}_n. \tag{71}$$

Note that $F_\mu(\alpha) \subset P_\mu$, so we have $p_\mu^2(\alpha)\lambda_\mu(\alpha) \leqslant c_\mu$. Now we see that the fidelity F^* given by expression (69) can only increase, when we increase coefficients $p_\mu(\alpha)$. Thus for any fixed c_μ it is optimal to choose $p_\mu(\alpha)$ satisfying

$$\forall \alpha \quad p_\mu^2(\alpha)\lambda_\mu(\alpha) = c_\mu. \tag{72}$$

Finally from the normalisation condition (expression (2) of (58)) and by substitution of (62) we get constraint on coefficients c_μ

$$\operatorname{Tr} X_A = \sum_\mu c_\mu \operatorname{Tr} P_\mu = \sum_\mu c_\mu d_\mu m_\mu = d^N. \tag{73}$$

Taking $v_\mu^2 = \frac{1}{d^N} c_\mu d_\mu m_\mu$ together with the equation ensuring maximal possible value of the quantity F^* given in (72) we write

$$p_\mu^2(\alpha)\lambda_\mu(\alpha) d_\mu m_\mu = \left(\frac{1}{d^N} c_\mu d_\mu m_\mu\right) d^N = d^N v_\mu^2. \tag{74}$$

Using the explicit formula for $\lambda_\mu(\alpha)$ we can compute $p_\mu(\alpha)$ in terms of new coefficients v_μ as

$$p_\mu(\alpha) = \frac{d^N}{\sqrt{N}} \sqrt{\frac{m_\alpha}{d_\alpha}} \frac{v_\mu}{m_\mu}. \tag{75}$$

Now inserting above formula into (69) we have

$$F^* = \frac{N}{d^{2N+2}} \max_{\{v_\mu\}} \sum_\alpha \frac{d_\alpha}{m_\alpha} \left(\sum_{\mu\in\alpha} \frac{d^N}{\sqrt{N}} \sqrt{\frac{m_\alpha}{d_\alpha}} \frac{v_\mu}{m_\mu} m_\mu\right)^2 = \frac{1}{d^2} \max_{\{v_\mu\}} \sum_\alpha \left(\sum_{\mu\in\alpha} v_\mu\right)^2. \tag{76}$$

Using equation (140) we get

$$d^N \left(\sum_\mu \frac{1}{d^N} c_\mu d_\mu m_\mu\right) = d^N \sum_\mu v_\mu^2 = d^N \Rightarrow \sum_\mu v_\mu^2 = 1. \tag{77}$$

The above condition is just a normalisation condition for some vector v, i.e. $||v||^2 = \sum_\mu v_\mu^2 = 1$. Finally writing more explicitly the double sum in (76) we see the following

$$\sum_\alpha \left(\sum_{\mu\in\alpha} v_\mu\right)^2 = \sum_\alpha \left(\sum_{\mu\in\alpha} v_\mu^2 + \sum_{\substack{\mu\neq\nu \\ \mu,\nu\in\alpha}} v_\mu v_\nu\right) = \sum_\mu n_\mu v_\mu^2 + \sum_{\substack{\mu\neq\nu \\ \mu/\nu=\square}} v_\mu v_\nu, \tag{78}$$

where n_μ is number of $\alpha \vdash N-1$ for which $\mu \in \alpha$. Having expression (78) together with (77) we rewrite the equation (76) as

$$F^* = \frac{1}{d^2} \max_{v:||v||=1} \langle v|M_F^d|v\rangle \equiv \frac{1}{d^2} ||M_F^d||_\infty, \tag{79}$$

$\square$

5.2. The dual SDP problem

The dual problem is to compute:

$$F_* = d^{N-2} \min_\Omega ||\operatorname{Tr}_B \Omega||_\infty, \tag{80}$$

with respect to constraints

$$\Omega - \sigma_a \geqslant 0, \quad a = 1, \ldots, N. \tag{81}$$

In the above Ω is an arbitrary operator acting on N subsystems. The solution of (80) with the constraints defined in (81) is given in the following

Theorem 31. *The quantity F_* in the dual problem can be expressed as:*

$$F_* = \frac{1}{d^2} ||M_F^d||_\infty, \tag{82}$$

where $||M_F^d||_\infty$ denotes the principal submatrix of the TM M_F which are given in definitions 10 and 3 respectively, and d denotes the dimension of the port.

Proof. Assume the general form of the operator which gives contribution to F_* as

$$\widetilde{\Omega} = \sum_{\alpha\vdash N-1} \widetilde{\Omega}(\alpha) = \sum_{\alpha\vdash N-1} \sum_{\mu\in\alpha} \omega_\mu(\alpha) F_\mu(\alpha), \quad \omega_\mu(\alpha) \geqslant 0. \tag{83}$$

By choosing coefficients $\omega_\mu(\alpha)$ we ensure that $\widetilde{\Omega} - \sigma_a \geqslant 0$ for $a = 1, \ldots, N$, where $\sigma_a = \frac{1}{d^{N-1}} \mathbf{1}_{\overline{an}} \otimes P^+_{a,n}$ (see condition (81)), and $P^+_{a,n}$ is projector onto the maximally entangled state $|\Phi^+\rangle_{a,n}$ between ath and nth subsystem. Due to symmetry it is enough to check it only for $a = N$, and on all irreps α.

$$\widetilde{\Omega} \geqslant \sigma_N \Longleftrightarrow \forall \alpha \quad \widetilde{\Omega}(\alpha) \geqslant \sigma_N P_\alpha, \tag{84}$$

where P_α denotes a Young projector onto irrep labelled by the Young diagram $\alpha \vdash N - 1$. More explicitly using form of the operator $\widetilde{\Omega}(\alpha)$ from (83) and resolution of the identity in terms of Young projectors P_α we have

$$\forall \alpha \vdash N - 1 \quad d^{N-1} \sum_{\mu \in \alpha} \omega_\mu(\alpha) F_\mu(\alpha) \geqslant P_\alpha \otimes P_+. \tag{85}$$

We now find the set of $\{\omega_\mu(\alpha)\}$ that satisfy the above inequality. Using generalisation desribed in [7] of theorem 1 and lemma 1 from [10] we know, that

$$A(\alpha) - \frac{1}{c(\alpha)} R(\alpha) \geqslant 0 \quad \text{if} \quad c(\alpha) = \langle \Phi^+ | \langle \varphi_{k,l}(\alpha) | A^{-1}(\alpha) | \varphi_{k,l}(\alpha) \rangle | \Phi^+ \rangle, \tag{86}$$

for all $1 \leqslant k \leqslant d_\alpha$, $1 \leqslant l \leqslant m_\alpha$. Plugging

$$A(\alpha) = d^{N-1} \sum_{\mu \in \alpha} \omega_\mu(\alpha) F_\mu(\alpha), \qquad R(\alpha) = P_\alpha \otimes P_+, \tag{87}$$

we are in the position to compute the constant $c(\alpha)$ for all irreps α

$$\begin{aligned} c(\alpha) &= \frac{1}{d^{N-1}} \sum_{k=1}^{d_\alpha} \sum_{l=1}^{m_\alpha} \langle \Phi^+ | \langle \varphi_{k,l}(\alpha) | \sum_{\mu \in \alpha} \omega_\mu^{-1}(\alpha) F_\mu(\alpha) | \varphi_{k,l}(\alpha) \rangle | \Phi^+ \rangle \\ &= \frac{1}{d^{N-1}} \sum_{\mu \in \alpha} \omega_\mu^{-1}(\alpha) \mathrm{Tr}[|\varphi_{k,l}(\alpha)\rangle \langle \varphi_{k,l}(\alpha)| \otimes P_+ F_\mu(\alpha)] \\ &= \frac{1}{d^N} \sum_{\mu \in \alpha} \omega_\mu^{-1}(\alpha) \frac{m_\mu}{m_\alpha}, \end{aligned} \tag{88}$$

since we used fact 39 from appendix A. One can see that because of fact 39 coefficient $c(\alpha)$ does not depend on indices k, l for all α. Now, redefining the operator $\widetilde{\Omega}(\alpha)$ as

$$\Omega(\alpha) \equiv c(\alpha) \widetilde{\Omega}(\alpha) = \frac{1}{d^N} \sum_{\nu \in \alpha} \omega_\nu^{-1}(\alpha) \frac{m_\nu}{m_\alpha} \sum_{\mu \in \alpha} \omega_\mu(\alpha) F_\mu(\alpha) = \frac{1}{d^N} \sum_{\nu \in \alpha} \sum_{\mu \in \alpha} \frac{m_\nu \omega_\mu(\alpha)}{m_\alpha \omega_\nu(\alpha)} F_\mu(\alpha) \tag{89}$$

we satisfy the constraint $\Omega - \sigma_N \geqslant 0$, since $\Omega = \sum_\alpha \Omega(\alpha)$. In the next step we compute the quantity $d^{N-2}\, \mathrm{Tr}_n \Omega$ from (80)

$$\begin{aligned} d^{N-2}\, \mathrm{Tr}_n \Omega &= \frac{1}{d^2} \sum_\alpha \sum_{\nu \in \alpha} \sum_{\mu \in \alpha} \frac{m_\nu \omega_\mu(\alpha)}{m_\alpha \omega_\nu(\alpha)} \mathrm{Tr}_n F_\mu(\alpha) = \frac{1}{d^2} \sum_\alpha \sum_{\nu \in \alpha} \sum_{\mu \in \alpha} \frac{m_\nu \omega_\mu(\alpha)}{m_\mu \omega_\nu(\alpha)} P_\mu \\ &= \frac{1}{d^2} \textstyle\sum_\alpha \displaystyle\sum_{\mu \in \alpha} \frac{\sum_{\nu \in \alpha} t_\nu(\alpha)}{t_\mu(\alpha)} P_\mu = \frac{1}{d^2} \sum_\mu \sum \times\, \alpha \in \mu \frac{\sum_{\nu \in \alpha} t_\nu(\alpha)}{t_\mu(\alpha)} P_\mu, \end{aligned} \tag{90}$$

where

$$t_\mu(\alpha) \equiv \frac{m_\mu}{\omega_\mu(\alpha)}. \tag{91}$$

From definition of $t_\mu(\alpha)$ we have to exclude all coefficients $\omega_\mu(\alpha)$ which are equal to zero from the decomposition (83). Finally, the quantity F_* in the dual problem given in (80) is given as

$$F_* = d^{N-2} \min_\Omega \| \mathrm{Tr}_n \Omega \|_\infty = \frac{1}{d^2} \min_{\{t_\mu(\alpha)\}} \max_\mu \sum_{\alpha \in \mu} \frac{\sum_{\nu \in \alpha} t_\nu(\alpha)}{t_\mu(\alpha)}. \tag{92}$$

Since we are looking for the feasible solution we assume that $\forall \alpha \;\; \forall \mu \in \alpha \;\; t_\mu(\alpha) = t_\mu$:

$$\forall \mu \vdash N \quad \sum_{\alpha \in \mu} \frac{\sum_{\nu \in \alpha} t_\nu}{t_\mu} = \frac{\sum_\nu (M_F^d)_{\mu\nu} t_\nu}{t_\mu}, \tag{93}$$

where matrix M_F^d is given in definition 3. Substituting (93) into (92) we reduce min–max problem to

$$F_* = \frac{1}{d^2} \min_{\{t_\mu\}} \max_\mu \frac{\sum_\nu (M_F^d)_{\mu\nu} t_\nu}{t_\mu}. \tag{94}$$

Consider the eigenproblem for the matrix $M_F^d t = \lambda t$, where $t = (t_\mu)$, and $\lambda \geqslant 0$, since M_F^d is positive semidefinite. Writing eigenproblem for M_F in the coordinates we have

$$\forall\,\mu\vdash N\quad \sum_{\nu}(M_F^d)_{\mu\nu}t_\nu=\lambda t_\mu\ \Rightarrow\ \lambda=\frac{\sum_\nu (M_F^d)_{\mu\nu}t_\nu}{t_\mu}.\tag{95}$$

Taking minimisation over all vectors t and maximal possible value over all allowed Young diagram μ we get definition of the maximal eigenvalue of the matrix M_F^d:

$$F_*=\frac{1}{d^2}\min_{\{t_\mu\}}\max_{\mu}\frac{\sum_\nu (M_F^d)_{\mu\nu}t_\nu}{t_\mu}=\frac{1}{d^2}||M_F^d||_\infty.\tag{96}$$

□

From theorems 30 and 31 we get:

Proposition 32.

- *From equality $F^*=F_*$ we find that*

$$F_{\text{opt}}=\frac{1}{d^2}||M_F^d||_\infty\tag{97}$$

is an optimal value of the fidelity in the case of the dPBT, where M_F^d is the principal submatrix of the TM M_F which are given in definitions 10 and 3 respectively, and d denotes the dimension of the port.

- *The optimal decorated POVMs $\Pi_a=\Pi\sigma_a\Pi$ for $a=1,\ldots,N$ where Π are given as:*

$$\Pi=\frac{d^N}{\sqrt{N}}\sum_{\alpha}\sum_{\mu\in\alpha}\sqrt{\frac{m_\alpha}{d_\alpha}}\frac{v_\mu}{m_\mu}F_\mu(\alpha),\tag{98}$$

where the σ_a is from (2). The coefficients v_μ are the components of the eigenvector v corresponding to the maximal eigenvalue of the M_F when $d\geqslant N$ or respective principal submatrix M_F^d otherwise.

- *The optimal resource state $|\Psi\rangle$:*

$$|\Psi\rangle=(O_A\otimes\mathbf{1}_B)|\psi^+\rangle_{A_1B_1}\otimes|\psi^+\rangle_{A_2B_2}\otimes\cdots\otimes|\psi^+\rangle_{A_NB_N},\tag{99}$$

where

$$O_A=\sqrt{d^N}\sum_\mu\frac{v_\mu}{\sqrt{d_\mu m_\mu}}P_\mu.\tag{100}$$

In the above P_μ denotes Young projector onto irrep labelled by the Young diagram $\mu\vdash N$.

Proof. Taking (61) together with (75) we obtain desired form of operator Π. To obtain expression (100) we use (62) with the condition $X_A=O_A^\dagger O_A$. □

In the regime $d\geqslant N$ from corollary 8 of section 4 we can give a simple formula for optimal fidelity F_{opt} in the dPBT:

$$F_{\text{opt}}=\frac{N}{d^2},\tag{101}$$

since in this particular case $||M_F^d||_\infty=||M_F||_\infty=N$. We can run the same analysis for the eigenvector $v=(v_\mu)$: when $d\geqslant N$ we know its analytical form as long as we assume that the respective characters of the irreps of $S(N)$ are given. In this case, such vector is given as a column of the reduced character matrix $T=(\chi_\mu(C))$ introduced in definition 6 of section 4. We can construct it explicitly due to item 4 in corollary 8. When $d<N$ we do not have analytical expressions (except for the qubit case discussed below) for the infinity norm of the principal submatrices of M_F or eigenvector v. In this case, we use the algorithm presented in the section 5.4.

The method of construction of the explicit matrix representation of the optimal POVMs and the state in the computational basis is described in detail in appendix C.

At the end of this section we discuss the asymptotic behaviour of the optimal fidelity $F_{\text{opt}}=F_{\text{opt}}(N,d)$ when number of ports N tends to infinity with fixed local dimension of the Hilbert space d. From corollary 13 and from well known relation $r(A)\leqslant||A||$, where $r(A)\equiv||A||_\infty$ is the spectral radius of $0\leqslant A=(a_{ij})\in\mathbb{M}(n,\mathbb{C})$, and $||\cdot||$ is any matrix norm we get that fidelity $F_{\text{opt}}(N,d)$ is bounded in the following way

$$\forall\,N,d\quad F_{\text{opt}}(N,d)\leqslant 1,\tag{102}$$

which certifies our calculations. Denote by $\widetilde{F}_{\text{ent}}=\widetilde{F}_{\text{ent}}(N,d)$ the lower bound for the fidelity in the non-optimised case, when the resource state is a tensor product of N d-dimensional singlets (see [1])

$$\widetilde{F}_{\text{ent}} = \frac{N}{d^2 + N - 1}. \tag{103}$$

We thus have $\widetilde{F}_{\text{ent}}(N, d) \leqslant F_{\text{opt}}(N, d)$. Moreover, for a fixed dimension d we have $\lim_{N\to\infty} \widetilde{F}_{\text{ent}}(N, d) = 1$, so together with expression (102) we see that $\lim_{N\to\infty} F_{\text{opt}}(N, d) = 1$.

5.3. Comparison with known results

In this section we compare our results to the only previously investigated case of $d = 2$ from [6–8]. We show how our approach relates to the latter when it comes to determining optimal fidelity and optimal decorated POVMs with the known representation of the dPBT. Moreover, we show how extending to higher dimensions of the underlying local Hilbert space reproduces the expression for the fidelity of the teleported state in the case of the maximally entangled resource state presented in [12]. The proof presented here, remarkably, does not require the notion of partially reduced irreps which was indispensable in the previous approach of [12].

We start from showing how the optimal fidelity F_{opt} given in proposition 32 from section 5.2 reduces to the results presented in earlier works. Whenever $N > 2,\ d = 2$ proposition 8 from section 4 is not applicable since not all irreps of $S(N)$ appear. We thus cannot use the analytical formula for the optimal fidelity given by 101 and instead have to carry out the analysis of the infinity norm of principal submatrices of M_F. Fortunately, for this case principal submatrices of M_F (we absorb coefficient $1/4$ into definition of M_F) reduce to so-called tridiagonal matrix of the form

$$M_F = \frac{1}{4}\begin{pmatrix} -x_1 + b & c & 0 & 0 & \cdots & 0 & 0 \\ a & b & c & 0 & \cdots & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & -x_2 + b \end{pmatrix} \in \mathbb{M}(t, \mathbb{R}), \tag{104}$$

for which analytical expressions for eigenvalues are known; t is the number of allowed Young diagrams of N for $d = 2,\ a = c = 1$, and $b = 2$. The coefficients x_1, x_2 depend on the parity of N. Let us consider them separately.

(a) $x_1 = 1$ and $x_2 = 0$ when N is odd.

In this case, from [15] (theorem 1, page 72) we know that all eigenvalues of M_F for $k = 1, \ldots, t$ are of the form:

$$\lambda_k = \frac{1}{4}\left[b + 2\sqrt{ac}\cos\left(\frac{2k\pi}{2t+1}\right)\right] = \frac{1}{2}\left[1 + \cos\left(\frac{2k\pi}{2t+1}\right)\right] = \cos^2\left(\frac{k\pi}{2t+1}\right), \tag{105}$$

since $\cos(2y) = 2\cos^2 y - 1$. When N is odd matrix M_F is $(N+1)/2-$ dimensional, so

$$\lambda_k = \cos^2\left(\frac{k\pi}{2\left(\frac{N+1}{2}\right) + 1}\right) = \cos^2\left(\frac{k\pi}{N+2}\right), \qquad k = 1, \ldots, (N+1)/2. \tag{106}$$

(a) $x_1 = x_2 = 1$ when N is even.

In this case, from [15] (theorem 4, page 73) we know that all eigenvalues of M_F for $k = 1, \ldots, t$ are of the form

$$\lambda_k = \frac{1}{4}\left[b + 2\sqrt{ac}\cos\left(\frac{k\pi}{t}\right)\right] = \frac{1}{2}\left[1 + \cos\left(\frac{k\pi}{t}\right)\right] = \cos^2\left(\frac{k\pi}{2t}\right). \tag{107}$$

When N is even matrix M_F is $N/2 + 1-$ dimensional, so

$$\lambda_k = \cos^2\left(\frac{k\pi}{2\left(\frac{N}{2} + 1\right)}\right) = \cos^2\left(\frac{k\pi}{N+2}\right), \qquad k = 1, \ldots, N/2 + 1. \tag{108}$$

In both cases, i.e. when N is odd or even the maximal eigenvalue is obtained for $k = 1$, and then optimal fidelity F_{opt} is equal to:

$$F_{\text{opt}} = ||M_F||_\infty = \cos^2\left(\frac{\pi}{N+2}\right). \tag{109}$$

We see that the above expression reproduces optimal fidelity in equation (41) from [8].

We now turn to the connection between our optimal decorated POVMs and those derived in [8] where authors propose the following optimal decorated POVMs

$$\widetilde{\Pi}_a = \sum_{s=s_{\min}}^{(N-1)/2} z(s)\rho(s)^{-1/y(s)}\sigma_a(s)\rho(s)^{-1/y(s)}, \quad a = 1, \ldots, N, \tag{110}$$

where s is the total spin number, and $z(s)$, $y(s)$ some constant numbers for fixed s. This expression is valid only for the qubit case, but it can be easily translate into language of the irreps of $S(N)$ and all $d \geqslant 2$. Assume the general form of the optimal decorated POVM to be

$$\widetilde{\Pi}_a = \sum_{\alpha\vdash N-1} z(\alpha)\rho(\alpha)^{-1/y(\alpha)}\sigma_a(\alpha)\rho(\alpha)^{-1/y(\alpha)}, \quad a = 1, \ldots, N, \tag{111}$$

where sum runs over all irreps labelled by Young diagrams of N whose height is not greater than dimension d of the underlying local Hilbert space. Now we are in the position to present direct connection between the most general decomposition of decorated POVMs presented in (60), (61) and the form given in (111).

Corollary 33. *Having decompositions of decorated POVMs defined in (60), (61), and (111) by comparison we can write the following equality between coefficients* $p_\mu(\alpha)$ *and* $z(\alpha)$*:*

$$p_\mu(\alpha) = \sqrt{z(\alpha)}\,\lambda_\mu(\alpha)^{-1/y(\alpha)}. \tag{112}$$

In particular for $d = 2$ *we have a direct translation between optimal decorated POVMs presented in [6–8] (or see (110)) and the decomposition presented in this manuscript.*

The equation (112) can be obtained by direct comparison of (60), (61) with the expression (111) and fact that $\rho = \sum_\alpha \sum_{\mu\in\alpha}\lambda_\mu(\alpha)F_\mu(\alpha)$.

Before we go further and prove that the choice of the decorated POVMs in (111) reproduces correct expression for the fidelity in the dPBT in the case of the maximally entangled resource state we need the following auxiliary lemma

Lemma 34. *The fidelity of the teleported state with the decorated POVMs given from (111) is given by*

$$F = \frac{N}{d^{2N}}\sum_{\alpha\vdash N-1} z(\alpha)d_\alpha m_\alpha c^2(\alpha, y(\alpha)), \tag{113}$$

where (see fact 39 and remark 40)

$$c(\alpha, y(\alpha)) = \frac{1}{d}\sum_{\mu\in\alpha}\lambda_\mu(\alpha)^{-1/y(\alpha)}\frac{m_\mu}{m_\alpha}. \tag{114}$$

Proof. From [8] we know that fidelity F in the deterministic version of the protocol is given by

$$F = \frac{1}{d^2}\mathrm{Tr}\left[\sum_{a=1}^{N}\Pi_a\sigma_a\right], \tag{115}$$

where Π_a are the POVMs given in (111). Using explicit form of POVMs we get:

$$\begin{aligned} F &= \frac{1}{d^2}\sum_{a=1}^{N}\mathrm{Tr}\left[\sum_\alpha z(\alpha)\rho(\alpha)^{-1/y(\alpha)}\sigma_a(\alpha)\rho(\alpha)^{-1/y(\alpha)}\sigma_a(\alpha)\right] \\ &= \frac{N}{d^{2N}}\sum_\alpha z(\alpha)\mathrm{Tr}[\rho(\alpha)^{-1/y(\alpha)}P_\alpha\otimes P_+\rho(\alpha)^{-1/y(\alpha)}P_\alpha\otimes P_+]. \end{aligned} \tag{116}$$

We used the fact that due to symmetry the trace in (115) does not depend on the index i and that $\sigma_N(\alpha) = P_\alpha \otimes P_+$. Using the decomposition of the Young projector $P_\alpha = \sum_{k=1}^{d_\alpha}\sum_{r=1}^{m_\alpha}|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)|$ we have

$$\begin{aligned} F &= \frac{N}{d^{2N}}\sum_\alpha z(\alpha)\sum_{k,l=1}^{d_\alpha}\sum_{r,p=1}^{m_\alpha}\mathrm{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)|\otimes P_+\rho(\alpha)^{-1/y(\alpha)}|\varphi_{l,p}(\alpha)\rangle\langle\varphi_{l,p}(\alpha)|\otimes P_+\rho(\alpha)^{-1/y(\alpha)}] \\ &= \frac{N}{d^{2N}}\sum_\alpha z(\alpha)\sum_{k,l=1}^{d_\alpha}\sum_{r,p=1}^{m_\alpha}\langle\varphi_{k,r}(\alpha)|\langle\Phi^+|\rho^{-1/y(\alpha)}|\varphi_{l,p}(\alpha)\rangle|\Phi^+\rangle\langle\varphi_{l,p}(\alpha)|\langle\Phi^+|\rho^{-1/y(\alpha)}|\varphi_{k,r}(\alpha)\rangle|\Phi^+\rangle. \end{aligned} \tag{117}$$

Using fact 39 and remark 40 we obtain statement of lemma. □

We do not claim yet that POVMs given by (111) are indeed the optimal ones for any $d \geqslant 2$. We only derived the formula for the fidelity of the teleported state for this particular choice of measurements. Now using above lemma we can show that

Lemma 35. *Substituting in expression (113) of lemma 34 and equation (110) $\forall\,\alpha\ y(\alpha)=2$ and $z(\alpha)=1$ we reproduce POVMs (square root measurement) and fidelity in the dPBT in the case of the maximally entangled state as a resource state.*

Proof. Inserting $\forall\,\alpha\ z(\alpha)=1,\ y(\alpha)=2$ into equation (111) we reproduce their form in the case of the maximally entangled state as a resource state. We get form of the square root measurement which we now is the optimal one in this case

$$\widetilde{\Pi}_a=\sum_{\alpha\vdash N-1}\frac{1}{\sqrt{\rho(\alpha)}}\sigma_a(\alpha)\frac{1}{\sqrt{\rho(\alpha)}},\qquad a=1,\dots,N. \tag{118}$$

Making the same substitution in equation (113) and using the explicit form of coefficients $c(\alpha,\ y(\alpha))$ given in remark 40 we get

$$F=\frac{N}{d^{2N+2}}\sum_{\alpha\vdash N-1}\frac{d_\alpha}{m_\alpha}\sum_{\mu,\mu'\in\alpha}\lambda_\mu(\alpha)^{-1/2}\lambda_{\mu'}(\alpha)^{-1/2}m_\mu m_{\mu'}. \tag{119}$$

Finally using explicit form of $\lambda_\mu(\alpha)=\frac{N}{d^N}\frac{m_\mu d_\alpha}{m_\alpha d_\mu}$ we have

$$F=\frac{1}{d^{N+2}}\sum_{\alpha\vdash N-1}\sum_{\mu',\mu\in\alpha}\sqrt{d_\mu m_\mu}\sqrt{d_{\mu'}m_{\mu'}}=\frac{1}{d^{N+2}}\sum_{\alpha\vdash N-1}\left(\sum_{\mu\in\alpha}\sqrt{d_\mu m_\mu}\right)^2. \tag{120}$$

We reproduce the formula for the fidelity of the teleported state from [12]. □

We can also reproduce expression for the fidelity of the teleported state in the case of the maximally entangled state using certain choice of the coefficients $p_\mu(\alpha)$ in the most general form of the POVM given by (61).

Corollary 36. *Choosing coefficients $p_\mu(\alpha)$ in the decomposition (61) as*

$$\forall\,\alpha\ \ \forall\,\mu\in\alpha\quad p_\mu(\alpha)=\frac{1}{\sqrt{\lambda_\mu(\alpha)}}=\sqrt{\frac{d^N}{N}\frac{m_\alpha d_\mu}{d_\alpha m_\mu}}, \tag{121}$$

and plugging them in (69) we reproduce fidelity for the maximally entangled state as a resource state (see theorem 12 of [12] or expression (120) above).

5.4. Convergent algorithm for computing fidelity

We now describe a method of approximation of maximal eigenvalues and the corresponding eigenvector of principal submatrices M_F^d [6]. We use this algorithm for $2<d<N$, since in this regime we do not know an analytical expressions for maximal eigenvalue and corresponding eigenvector of the matrix M_F^d which are required for computation of F_{opt} together with optimal state and decorated POVMs, but of course it works for all values of d and N. From fact 14 and corollary 15 from section 4 we can apply Frobenius–Perron theorem (see theorem 44 of appendix B) to M_F as well as to all of its principal submatrices M_F^d, and write

Proposition 37. *If matrix $A\in\mathbb{M}(n,\ \mathbb{R})$ is non-negative and irreducible then it satisfies the following eigenequation*

$$Ax=r(A)x, \tag{122}$$

where $x=(x_i):\sum_i x_i=1$ and $x_i>0$, so this eigenvector is positive. Such a vector x is called Perron eigenvector of the matrix A.

Making use of irreducibility and the primitivity, one can approximate maximum eigenvalues and find the corresponding eigenvector of M_F^d, which are positive semidefinite and primitive (see corollary 8 and remark 11).

Theorem 38. *Let $A\in\mathbb{M}(n,\ \mathbb{R})$ be a positive semidefinite and primitive matrix (in particular M_F^d). Suppose that the vector w^0 is of the form*

$$w^0=(w_i^0):\sum_i^n w_i^0=1,\quad w_i^0>0, \tag{123}$$

[6] SageMath code for implementing the algorithm as well as routines to generate the respective matrices is available upon request.

then we define

$$v^{m+1} = Aw^m, \quad m = 0, 1, \dots \qquad w^{m+1} = \frac{v^{m+1}}{\sum_j v_j^{m+1}}, \quad m = 0, 1, \dots \tag{124}$$

We thus have the following limits

$$\lim_{m\to\infty} w^m = w, \qquad \lim_{m\to\infty} \sum_j^n v_j^m = r(A), \tag{125}$$

where w is Perron eigenvector of the matrix A. So the sequence of vectors $\{w^m\}$ approximates Perron eigenvector of the matrix A, whereas the number sequence $\{\sum_j^n v_j^m\}$ approximates the spectral radius $r(A)$ of the matrix A.

Proof. We use the method of calculation of eigenvalues of diagonalizable matrices described in [9], and for sake of completeness of this manuscript, we adopt this method to our particular case of positive semidefinite, non-negative and irreducible matrices.

By induction using the non-negativity and irreducibility of the matrix A we get

$$\forall\, m \in \mathbb{N} \quad v^m = (v_i^m) : v_i^m > 0 \Rightarrow \sum_j v_j^m > 0, \tag{126}$$

so the vectors w^m are well defined. From our assumptions on the matrix A and Perron–Frobenius theorem it follows that A has the following spectral decomposition

$$A = \sum_{k=1}^K \mu_k P_k, \tag{127}$$

where $\mu_1 = r(A) > \mu_t$: $t \geqslant 2$ and $P_1 = p_1 p_1^\dagger$: $p_1 = \frac{w}{\| w \|} \in \mathbb{R}^n$. The vector w is the Perron vector of the matrix A, so it satisfies $w = (w_i)$: $\sum_i w_i = 1$, $w_i > 0$. The remaining projectors have the form the standard form

$$P_k = \sum_l p_k^l p_k^{l\dagger} : p_k^l = (p_{ki}^l) \in \mathbb{R}^n, \quad \|p_k^l\| = 1, \quad k \geqslant 2. \tag{128}$$

Using this spectral decomposition we calculate

$$v^1 = (v_i^1) = \mu_1 p_1 (p_1, w^0) + \sum_{k\geqslant 2} \mu_k \sum_l p_k^l (p_k^l, w^0), \tag{129}$$

where (p_1, w^0) is the standard, Euclidean scalar product of vectors in the space $\mathbb{R}^n$. From this we get

$$\sum_j v_j^1 = \mu_1 s(p_1)(p_1, w^0) + \sum_{k\geqslant 2} \mu_k \sum_l s(p_k^l)(p_k^l, w^0), \tag{130}$$

where $s(x) = \sum_{i=1} x_i$ for $x = (x_i) \in \mathbb{R}^n$. So we have

$$w^1 = \frac{\mu_1 p_1 (p_1, w^0) + \sum_{k\geqslant 2} \mu_k \sum_l p_k^l (p_k^l, w^0)}{\mu_1 s(p_1)(p_1, w^0) + \sum_{k\geqslant 2} \mu_k \sum_l s(p_k^l)(p_k^l, w^0)}. \tag{131}$$

By induction we get

$$w^m = \frac{\mu_1^m p_1 (p_1, w^0) + \sum_{k\geqslant 2} \mu_k^m \sum_l p_k^l (p_k^l, w^0)}{\mu_1^m s(p_1)(p_1, w^0) + \sum_{k\geqslant 2} \mu_k^m \sum_l s(p_k^l)(p_k^l, w^0)} \tag{132}$$

and

$$\sum_{j=1}^n v_j^{m+1} = \frac{\mu_1^{m+1} s(p_1)(p_1, w^0) + \sum_{k\geqslant 2} \mu_k^{m+1} \sum_l s(p_k^l)(p_k^l, w^0)}{\mu_1^m s(p_1)(p_1, w^0) + \sum_{k\geqslant 2} \mu_k^m \sum_l s(p_k^l)(p_k^l, w^0)}, \tag{133}$$

where $\mu_1 = r(A) > \mu_t$: $t \geqslant 2$ and $s(p_1) = \frac{\sum_i w_i}{\| w \|} = \frac{1}{\| w \|} > 0$, $(p_1, w^0) > 0$. We thus have

$$\lim_{m\to\infty} w^m = \frac{p_1}{s(p_1)} = w, \qquad \lim_{m\to\infty} \sum_{j=1}^n v_j^{m+1} = \mu_1 = r(A). \tag{134}$$

□

6. Conclusions and discussion

We showed that the question of optimal functioning of the dPBT can be reduced to finding a maximal eigenvalue of a certain class of matrices which encode the relationship between Young diagrams. Remarkably, this teleportation protocol can be fully characterised in terms of a single 'static' object—the TM. This brings about a question on whether one could reduce the study of the optimal performance of other important LOCC protocols in Quantum Information Processing to a study of a simple object which encodes the relationship between the given input and the desired output states of such a protocol analogously do the dPBT.

Acknowledgements

We would like to thank anonymous Referees for valuable comments which improved presentation of the results. MS is supported by the grant 'Mobilność Plus IV', 1271/MOB/IV/2015/0 from the Polish Ministry of Science and Higher Education. MH and MM are supported by National Science Centre, Poland, grant OPUS 9. 2015/17/B/ST2/01945.

Appendix A. Auxiliary facts and lemmas

The set of vectors $\{|\varphi_{k,r}(\alpha)\rangle\}_{k=1}^{d_\alpha}$ spans the rth irrep of $S(N-1)$ is labelled by Young diagram α. Define the following operators

$$E_{kl}^{\alpha}=\sum_{r=1}^{m_\alpha}|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,r}(\alpha)|, \tag{135}$$

where m_α is a multiplicity of irrep labelled by α. The above operators play an important role in the description of the irreps of the symmetric group, but we skip the details here (see for example appendix F of [12]).

Fact 39. Assume, that $F_\mu(\alpha)$ are projectors onto irreps of algebra $\mathcal{A}_n^{t_n}(d)$, then

$$\langle\varphi_{k,r}(\alpha)|\langle\Phi^+|F_\mu(\alpha)|\Phi^+\rangle|\varphi_{l,s}(\alpha)\rangle=\frac{1}{d}\frac{m_\mu}{m_\alpha}\delta_{kl}\delta_{rs}, \tag{136}$$

where vectors $\{|\varphi_{k,r}(\alpha)\rangle\}_{k=1}^{d_\alpha}$ span the rth irrep of $S(N-1)$ labelled by Young diagram α, and $|\Phi^+\rangle\langle\Phi^+|$ is maximally entangled state between two last subsystems.

Proof. To prove statement first we use $|\Phi^+\rangle\langle\Phi^+|=\frac{1}{d}V^{t_n}(N,n)$, property $F_\mu(\alpha)=M_\alpha P_\mu$ (see theorem 1 from [12]), and $V^{t_n}(N,n)M_\alpha=V^{t_n}(N,n)P_\alpha$ (see fact 13 from [12])

$$\begin{aligned}\langle\varphi_{k,r}(\alpha)|\langle\Phi^+|F_\mu(\alpha)|\Phi^+\rangle|\varphi_{l,s}(\alpha)\rangle&=\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|\otimes|\Phi^+\rangle\langle\Phi^+|F_\mu(\alpha)]\\&=\frac{1}{d}\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|V^{t_n}(N,n)M_\alpha P_\mu]=\frac{1}{d}\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|V^{t_n}(N,n)P_\alpha P_\mu]\\&=\frac{1}{d}\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|P_\alpha V^{t_n}(N,n)P_\mu],\end{aligned} \tag{137}$$

since $[V^{t_n}(N,n),P_\alpha]=0$. In the next step having decomposition $P_\alpha=\sum_{k=1}^{d_\alpha}\sum_{r=1}^{m_\alpha}|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{k,r}(\alpha)|$ and taking into account that the set $\{|\varphi_{k,r}(\alpha)\rangle\}_{k=1}^{d_\alpha}$ is formed of the orthonormal vectors we conclude, that $|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|P_\alpha=|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|$. Having this we write

$$\begin{aligned}\langle\varphi_{k,r}(\alpha)|\langle\Phi^+|F_\mu(\alpha)|\Phi^+\rangle|\varphi_{l,s}(\alpha)\rangle&=\frac{1}{d}\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|V^{t_n}(N,n)P_\mu]\\&=\frac{1}{d}\operatorname{Tr}[|\varphi_{k,r}(\alpha)\rangle\langle\varphi_{l,s}(\alpha)|P_\mu],\end{aligned} \tag{138}$$

because $\operatorname{Tr}_n V^{t_n}(N,n)=\mathbf{1}$, where $\mathbf{1}$ is the identity operator on the second last subsystem. Using Fact 19 from [12] we have the following decomposition of P_μ

$$P_\mu=\frac{d_\mu}{(n-1)!}\sum_{a=1}^{n-1}\sum_{i,j=1}^{d_\mu}\varphi_{ij}^{\mu}(a,n-1)V(a,n-1)\sum_{\pi\in S(n-2)}\varphi_{ji}^{\mu}(\pi^{-1})V(\pi). \tag{139}$$

Substituting (139) into (138) and using property $\operatorname{Tr}_{n-1}V(a,n-1)=d^{\delta_{a,n-1}}\mathbf{1}$, where identity operator $\mathbf{1}$ acts on the first $n-1$ subsystems we reduce to right-hand side to

$$\begin{aligned}&\frac{d_\mu}{d(n-1)!}\sum_{a=1}^{n-1}d^{\delta_{a,n-1}}\sum_{i,j=1}^{d_\mu}\varphi_{ij}^{\mu}(a,n-1)\sum_{\pi\in S(n-2)}\varphi_{ji}^{\mu}(\pi^{-1})\langle\varphi_{l,s}(\alpha)|V(\pi)|\varphi_{k,r}(\alpha)\rangle\\&=\frac{d_\mu}{d(n-1)!}\sum_{a=1}^{n-1}d^{\delta_{a,n-1}}\sum_{i,j=1}^{d_\mu}\varphi_{ij}^{\mu}(a,n-1)\sum_{\pi\in S(n-2)}\varphi_{ji}^{\mu}(\pi^{-1})\varphi_{lk}^{\alpha}(\pi)\delta_{rs}.\end{aligned} \tag{140}$$

Writing element $\varphi^{\mu}_{ji}(\pi^{-1})$ in the PRIR basis (see appendix B of [12]) as $\sum_{\beta\in\mu}\sum_{i_\beta,j_\beta=1}^{d_\beta}(\varphi^{\mu}_{R})^{\beta\beta}_{j_\beta i_\beta}(\pi^{-1})$ we have

$$\sum_{\pi\in S(n-2)}\varphi^{\mu}_{ji}(\pi^{-1})\varphi^{\alpha}_{lk}(\pi)=\sum_{\beta\in\mu}\sum_{i_\beta,j_\beta=1}^{d_\beta}\left(\sum_{\pi\in S(n-2)}(\varphi^{\mu}_{R})^{\beta\beta}_{j_\beta i_\beta}(\pi^{-1})\varphi^{\alpha}_{lk}(\pi)\right) \tag{141}$$

Substituting above to (140) and using standard orthogonality relation for irreps $\sum_{\sigma\in S(n)}\varphi^{\alpha}_{ij}(\sigma)\varphi^{\beta}_{kl}(\sigma^{-1})=\frac{n!}{d_\alpha}\delta^{\alpha\beta}\delta_{jk}\delta_{il}$ (see for example [4]) we obtain

$$\langle\varphi_{k,r}(\alpha)|\langle\Phi^+|F_\mu(\alpha)|\Phi^+\rangle|\varphi_{l,s}(\alpha)\rangle=\frac{1}{d(n-1)}\frac{d_\mu}{d_\alpha}\left(\sum_{a=1}^{n-1}d^{\delta_{a,n-1}}(\varphi^{\mu}_{R})^{\alpha\alpha}_{lk}(a,n-1)\right)\delta_{rs}. \tag{142}$$

The expression in bracket is just proportional to eigenvalues of PBT operator multiplied by δ_{lk} (see appendix D.2 of [12]), namely is equal to $(n-1)\frac{m_\mu d_\alpha}{m_\alpha d_\mu}\delta_{lk}$, which gives desired result. □

Remark 40. As a natural consequence of fact 39 we have for $k,l=1,\ldots,d_\alpha$ and $r,p=1,\ldots,m_\alpha$ the following

$$\langle\Phi^+|\langle\varphi_{k,r}(\alpha)|\rho^{-1/y(\alpha)}|\Phi^+\rangle|\varphi_{l,p}(\alpha)\rangle=c(\alpha,y(\alpha))\delta_{kl}\delta_{rp}, \tag{143}$$

where

$$c(\alpha,y(\alpha))\equiv\frac{1}{d}\sum_{\mu\in\alpha}\lambda_\mu(\alpha)^{-1/y(\alpha)}\frac{m_\mu}{m_\alpha}, \tag{144}$$

and $y(\alpha)$ is an arbitrary non-zero real number depending on Young diagram $\alpha\vdash N-1$.

Using that $\rho(\alpha)=\sum_{\mu\in\alpha}\lambda_\mu(\alpha)F_\mu(\alpha)$ we get the desired statement.

Appendix B. Additional facts from general matrix theory

We begin with a short overview of some basic facts from the matrix theory which are required for the analysis of the spectral properties of the matrix M_F described in section 4. We discuss the notion of irreducibility for the matrices with non-negative entries (which is the case for matrix M_F) and primitivity.

Recall Sylwester's theorem [5]

Theorem 41 (Sylwester). *A Hermitian matrix A is positive semidefinite if and only if all principal minors are positive.*

Definition 42. Let $A\in\mathbb{M}(m,\mathbb{C})$, then the matrix A is irreducible if it cannot be conjugated into the block upper triangular form by a permutation matrix P:

$$PAP^{-1}\neq\begin{pmatrix}A_1 & A_2\\ 0 & A_3\end{pmatrix}, \tag{145}$$

where A_1, A_3 are non-trivial square matrices.

If $A\in\mathbb{M}(m,\mathbb{R})$ is non-negative, then we have an equivalent definition (which is the case for the TM M_F):

Definition 43. Let $A\in\mathbb{M}(m,\mathbb{R})$ be a non-negative matrix, then the matrix A is irreducible if for any pair of indices $1\leqslant i,j\leqslant n$ there exists a $q\in\mathbb{N}$ such that $(A^q)_{ij}>0$.

We now present a stronger version of the Frobenius–Perron theorem:

Theorem 44 (Frobenius–Perron). *Let A be an $m\times m$ irreducible matrix with non-negative, real entries with the spectral radius $r(A)$. Then we have the following:*

1. *The number $r(A)$ is a positive real number and it is an eigenvalue of matrix A (Perron–Frobenius eigenvalue).*
2. *The multiplicity of an eigenvalue $r(A)$ is equal to one.*
3. *The matrix A has an eigenvector corresponding to an eigenvalue r with all positive components.*

Definition 45. A non-negative matrix $A\in\mathbb{M}(m,\mathbb{R})$ is primitive if it is irreducible and has only one non-zero eigenvalue of maximum modulus.

On the other hand we have [5]:

Proposition 46. *If the matrix* $A \in \mathbb{M}(m, \mathbb{R})$ *is non-negative, irreducible, and has positive diagonal then* A *is primitive.*

At the end we introduce the notion of centrosymmetric matrices.

Definition 47. Matrix $A \in \mathbb{M}(m, \mathbb{C})$ is called centrosymmetric if its entries satisfy

$$A_{i,j} = A_{m-i+1,m-j+1} \quad \text{for} \quad 1 \leqslant i, j \leqslant m. \tag{146}$$

Appendix C. The explicit form of Young projectors and operators $F_\mu(\alpha)$ in natural representation

We provide the construction of the permutation operators $V(\sigma)$, where $\sigma \in S(N)$, Young projectors P_μ, and projectors $F_\mu(\alpha)$ in the computational basis. Using this representation we can construct the explicit form of the optimal decorated POVM (98) and state (99) for various N, d.

Consider a unitary representation of a permutation group $S(N)$ acting on the N-fold tensor product of complex spaces $\mathbb{C}^d$, so our full Hilbert space is $\mathcal{H} \cong (\mathbb{C}^d)^{\otimes N}$. For a fixed permutation $\sigma \in S(N)$ a unitary transformation $\mathrm{V}(\sigma)$ is given by

$$V(\sigma)(|e_{i_1}\rangle \otimes \ldots \otimes |e_{i_N}\rangle) = |e_{i_{\sigma^{-1}(1)}}\rangle \otimes \ldots \otimes |e_{i_{\sigma^{-1}(N)}}\rangle, \tag{147}$$

where the set $\{|e_{i_1}\rangle \otimes \cdots \otimes |e_{i_N}\rangle\}$ is a standard basis in $(\mathbb{C}^d)^{\otimes N}$. Then, the explicit form of the operator $V(\sigma)$ for some $\sigma \in S(N)$ is given by

$$V(\sigma) = \sum_{e_{i_1}, \ldots, e_{i_N}} |e_{i_{\sigma^{-1}(1)}}\rangle \otimes \cdots \otimes |e_{i_{\sigma^{-1}(N)}}\rangle \langle e_{i_1}| \otimes \cdots \otimes \langle e_{i_N}|. \tag{148}$$

Using an expression for any permutation operator $V(\sigma)$, the explicit form of Young projectors in the natural representation is

$$P_\mu = \frac{f_\mu}{N!} \sum_{\sigma \in S(N)} \chi^\mu(\sigma^{-1}) V(\sigma), \tag{149}$$

where $\chi^\mu(\sigma)$ is the character calculated on the irrep labelled by the Young diagram $\mu \vdash N$ on the permutation $\sigma \in S(N)$, f^μ is some constant depending on the Young diagram $\mu \vdash N$ (see for example [4]). The explicit form of the projectors $F_\mu(\alpha)$ described briefly in the introductory part of our manuscript (for complete description see [12]) are given by

$$F_\mu(\alpha) = \frac{1}{\gamma_\mu(\alpha)} P_\mu \sum_{a=1}^{N} V(a, N) P_\alpha \otimes \widetilde{P}_+ V(a, N) P_\mu, \tag{150}$$

where P_α, P_μ are Young projectors onto irreducible spaces labelled by Young diagrams $\alpha \vdash N - 1$ and $\mu \vdash N - 1$ respectively, $\widetilde{P}_+$ is an unnormalised projector onto the maximally entangled state between Nth and $n = N + 1$th, and $\gamma_\mu(\alpha)$ is given in (8).

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